NATURAL SCIENCES TRIPOS Part IB & II (General)

Tuesday, 26 May, 2015  9:00 am to 12:00 pm

MATHEMATICS (1)

Before you begin read these instructions carefully:

You may submit answers to no more than six questions. All questions carry the same number of marks.

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on one side of the paper only and begin each answer on a separate sheet.

At the end of the examination:

Each question has a number and a letter (for example, 6A).

Answers must be tied up in separate bundles, marked A, B or C according to the letter affixed to each question.

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A separate green master cover sheet listing all the questions attempted must also be completed.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Green master cover sheet
Script paper

SPECIAL REQUIREMENTS
Calculator - students are permitted to bring an approved calculator.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.
Consider a toroidal body defined parametrically in the Cartesian coordinate system \( \mathbf{x} = (x, y, z) \) as the region satisfying

\[
\begin{align*}
x &= (1 + r \sin \alpha) \cos \beta, \\
y &= (1 + r \sin \alpha) \sin \beta, \\
z &= r \cos \alpha,
\end{align*}
\]

with \( 0 \leq r \leq R, -\pi \leq \alpha < \pi \) and \( 0 \leq \beta < 2\pi \) for constant \( 0 < R < 1 \).

(a) For a toroidal coordinate system \((r, \alpha, \beta)\), determine the Cartesian components of the vectors \( \mathbf{h}_r, \mathbf{h}_\alpha, \mathbf{h}_\beta \) such that the Cartesian differential \( d\mathbf{x} \) is given by

\[
d\mathbf{x} = h_r dr + h_\alpha d\alpha + h_\beta d\beta,
\]

and hence establish whether or not the toroidal coordinate system is orthogonal. Determine the Jacobian for the coordinate transformation. \([8]\)

(b) Suppose the toroidal body is immersed in a vector field \( \mathbf{F}(\mathbf{x}) = \nabla \Omega + \nabla \times \mathbf{U} \), where \( \Omega \) is a scalar field and \( \mathbf{U} \) is a vector field. Consider the integral

\[
I = \int_S \mathbf{F} \cdot d\mathbf{S},
\]

where \( S \) is the surface of the body and \( d\mathbf{S} \) is an element of vector area. Why does \( I \) not depend on \( \mathbf{U} \)? \([3]\)

(c) Determine \( I \) for the case \( \Omega = z^4 - xyz + e^{-3y} \cos 3x + e^{2x} \sin 2y \). \([9]\)
In two spatial dimensions the time evolution of a scalar field \( u(x, y, t) \) is given by

\[
\frac{\partial u}{\partial t} = \nabla^2 u + \beta \frac{\partial u}{\partial x},
\]

for constant \( \beta \).

(a) Consider the domain \( 0 \leq x \leq 1, \ 0 \leq y \leq 1 \) with boundary conditions \( \frac{\partial u}{\partial y} = 0 \) on \( y = 0 \) and \( u = 0 \) on the other three boundaries. Use separation of variables to determine the general solution of (*) satisfying the boundary conditions and show that at \( t = 0 \) this reduces to

\[
u(x, y, 0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} e^{-\frac{\beta}{x} \sin(m\pi x) \cos\left(\frac{2n-1}{2}\pi y\right)},
\]

for some set of constants \( A_{mn} \). \[14\]

(b) Determine the constants \( A_{mn} \) required for \( u \) to satisfy the initial condition

\[
u(x, y, 0) = x(1-x)e^{-\frac{\beta}{x} \cos\left(\frac{3}{2}\pi y\right)}.
\]

\[6\]
(a) Use the method of Green’s functions to solve

\[ \frac{d^2 y}{dx^2} - y = f(x) \]

for \( 0 \leq x \leq 1 \), with the boundary conditions \( y(0) = y(1) = 0 \). \[\text{[9]}\]

[You may use the identity \( \sinh a \cosh b - \cosh a \sinh b = \sinh(a - b) \).]

(b) A forced damped harmonic oscillator satisfies the equation

\[ \frac{d^2 x}{dt^2} + 2 \frac{dx}{dt} + (1 + \mu^2) x = g(t), \]

where \( \mu \) is a positive constant.

(i) Solve this equation for \( t \geq 0 \) with initial conditions \( x = dx/dt = 0 \) at \( t = 0 \). \[\text{[7]}\]

(ii) Assume that \( |g(t)| < Ce^{-at} \) where \( C \) and \( a \) are constants with \( a > 0 \). Prove that \( x(t) \to 0 \) as \( t \to \infty \). \[\text{[4]}\]
A radio station wishes to analyse its broadcast of the signal \( a(t) \) using Fourier analysis. As a test signal, the radio station chooses a single pulse given by

\[
a(t) = \begin{cases} 
1 & \text{if } |t| < 1, \\
0 & \text{otherwise}.
\end{cases}
\]

(a) Determine \( \tilde{a}(\omega) \), the Fourier transform of \( a(t) \).

(b) Due to bandwidth limitations, the radio station decides to filter the signal so that it broadcasts

\[
b(t) = \int_{-\infty}^{\infty} a(s) f(t-s) \, ds,
\]

where the filter \( f(t) \) is defined as

\[
f(t) = \begin{cases} 
1 - |t| & \text{if } |t| < 1, \\
0 & \text{otherwise}.
\end{cases}
\]

Determine \( \tilde{f}(\omega) \), the Fourier transform of \( f(t) \), and hence use the convolution theorem to derive an expression for \( \tilde{b}(\omega) \), the Fourier transform of \( b(t) \).

(c) Due to reflections from a mountain range, the signal received by the listeners can be modelled by

\[
r(t) = b(t) + \alpha b(t - \tau),
\]

where \( \alpha \) is the relative strength of the reflected signal and \( \tau \) is the delay in receiving the reflection. Determine \( \tilde{r}(\omega) \), the Fourier transform of \( r(t) \).

(d) Measurements of the received signal suggest it is well approximated by \( s(t) \) with Fourier transform

\[
\tilde{s}(\omega) = 2 \left(1 + \epsilon \omega^2\right) e^{-\omega^2/4}
\]

for some constant \( \epsilon \). Determine \( s(t) \).

[You may assume \( \int_{-\infty}^{\infty} e^{-(z-ia)^2} \, dz = \sqrt{\pi} \) for real \( a \).]
(a) Let \( A \) and \( B \) be \( n \times n \) Hermitian matrices that commute, i.e., \( AB = BA \). Assuming that the eigenvalues of \( A \) and \( B \) are non-degenerate, show that the eigenvectors of \( A \) and \( B \) are the same so that \( A \) and \( B \) may be written as \( A = U \Lambda_A U^\dagger \) and \( B = U \Lambda_B U^\dagger \), where \( \Lambda_A \) and \( \Lambda_B \) are diagonal matrices and \( U \) is unitary. \[4\]

For such matrices \( A \) and \( B \), using this result, or otherwise, show that

\[
\exp(A) \exp(B) = \exp(A + B),
\]

where the exponential of a square matrix \( A \) is defined by the series

\[
\exp(A) = I + A + \frac{1}{2!} A^2 + \frac{1}{3!} A^3 + \cdots,
\]

with \( I \) the identity matrix. \[4\]

(b) For general \( n \times n \) matrices \( X \) and \( Y \), verify that

\[
\exp(\epsilon X) \exp(\epsilon Y) = \exp \left( \epsilon X + \epsilon Y + \frac{1}{2} \epsilon^2 [X, Y] \right) + O(\epsilon^3),
\]

where \( \epsilon \) is an arbitrary parameter and

\[ [X, Y] \equiv XY - YX \]

(c) For the matrix

\[
M = \begin{pmatrix}
0 & a & 0 & 0 \\
a & 0 & 0 & 0 \\
0 & 0 & 0 & b \\
0 & 0 & -b & 0
\end{pmatrix},
\]

where \( a \) and \( b \) are real, show that

\[
\exp(M) = \begin{pmatrix}
\cosh a & \sinh a & 0 & 0 \\
\sinh a & \cosh a & 0 & 0 \\
0 & 0 & \cos b & \sin b \\
0 & 0 & -\sin b & \cos b
\end{pmatrix}.
\]

\[7\]
6A

(a) Explain how to diagonalize a real symmetric matrix $A$. [4]

(b) Describe the quadratic surface $\Sigma$ in $\mathbb{R}^3$ defined by

$$5x_1^2 - 8x_1x_2 + 5x_2^2 + 9x_3^2 = 9,$$

specifying the principal axes and, where appropriate, the semi-axis lengths. [6]

Show that $\Sigma$ intersects the surface defined by

$$x_1^2 + x_2^2 + x_3^2 = 4$$

in a pair of circles, and find their orientations, radii, and centres. [5]

(c) On a general quadratic surface defined by $x^TAx = 1$, with $A$ a real symmetric matrix, show that the squared distance from the origin, $x^Tx$, is extremised for $x$ an eigenvector of $A$. [5]

7B

(a) Derive the Cauchy–Riemann equations for the analytic function

$$f(z) = u(x, y) + iv(x, y),$$

where $z = x + iy$. [2]

(b) Determine the analytic function $f(z)$ if $u(x, y) = x \cos x \cosh y + y \sin x \sinh y$. [7]

(c) Assume that $g(z)$ is analytic and $|g(z)|$ is constant. Prove that $g(z)$ is constant. [6]

(d) Calculate the Taylor series of the function

$$h(z) = \frac{2z}{z^2 + 1}$$

about $z = 1$ and state its radius of convergence. [5]

[Hint: use partial fractions.]
8B

Consider the equation

\[(1 - x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + n^2 y = 0.\]  \((\star)\)

(a) Show that \(x = 0\) is an ordinary point, and determine the nature of the points \(x = \pm 1\) and \(x = \infty\). \([6]\)

(b) Explain how to construct two independent solutions of \((\star)\) as power series about \(x = 0\). What is the recurrence relation for the coefficients of these series? \([6]\)

(c) Use the ratio test to determine the radius of convergence of each series. How are these related to the location of the singular points of \((\star)\)? \([3]\)

(d) Show that polynomial solutions of \((\star)\) exist when \(n\) is an integer. With the condition \(y(1) = 1\), determine these solutions for the cases \(n = 0, 1, 2,\) and \(3\). \([5]\)
(a) State the Euler–Lagrange equation that determines the extrema of the functional $F[z]$ of the function $z(x)$, where

$$F[z] = \int_{\alpha}^{\beta} f(z, z'; x) \, dx,$$
with primes denoting differentiation with respect to $x$. \[2\]

If $f$ does not depend explicitly on $x$, show that

$$f - z' \frac{\partial f}{\partial z'} = \text{const.}$$

when $z(x)$ extremises $F[z]$. \[4\]

Explain how to determine the extrema of $F$ subject to the constraint that a further functional $G[z]$ is constant. \[2\]

(b) An inextensible string of total length $\pi a/2$ hangs under its own weight in the $x$–$z$ plane, with its endpoints fixed at $z = 0$ and $x = \pm a/\sqrt{2}$. The mass per unit length of the string, $\mu$, is uniform. The gravitational potential $\Phi(z)$ varies with $z$, and is defined such that the potential energy of an element of the string of mass $\delta m$ is $\delta m \Phi$. Parameterising the path of the string as $z(x)$, show that the total gravitational potential energy is

$$V[z] = \mu \int_{-a/\sqrt{2}}^{a/\sqrt{2}} \Phi(z) \sqrt{1 + z'^2} \, dx.$$ \[2\]

The shape adopted by the string is such as to minimise $V$ subject to the constraint of a fixed length. Show that $z(x)$ satisfies

$$\frac{\mu \Phi(z) - \lambda}{\sqrt{1 + z'^2}} = \text{const.},$$

where $\lambda$ is a constant. \[5\]

Determine a suitable $\Phi(z)$ if the string is to hang along an arc of a circle of radius $a$. \[5\]
(a) Consider the functionals
\[ F[y] = \int_{\alpha}^{\beta} \left[ p(x) (y')^2 + q(x) y^2 \right] \, dx, \quad G[y] = \int_{\alpha}^{\beta} w(x) y^2 \, dx, \]
where \( p(x) > 0, \, q(x) \geq 0, \) and \( w(x) > 0 \) for \( \alpha < x < \beta, \) and primes denote differentiation with respect to \( x. \) Show that if suitable boundary conditions are imposed at \( x = \alpha \) and \( x = \beta, \) the ratio \( F[y]/G[y] \) is extremised when \( y \) satisfies the Sturm–Liouville eigenvalue equation
\[
- \left[ p(x) y' \right]' + q(x) y = \lambda w(x) y. \tag{*}
\]
How do the eigenvalues \( \lambda \) relate to the extremal values of \( F[y]/G[y]? \)
Hence explain the Rayleigh–Ritz method for estimating the lowest eigenvalue of \( (*) \).

(b) Pressure waves in a spherical cavity of radius \( a \) satisfy
\[
\nabla^2 \psi + k^2 \psi = 0, \tag{†}
\]
with \( k \) a real constant. The function \( \psi \) is bounded everywhere and vanishes on \( r = a. \)

(i) For spherically-symmetric solutions \( \psi(r) \), show that \( (†) \) reduces to an ordinary differential equation that can be written in Sturm–Liouville form with \( p(r) = r^2, \, q(r) = 0, \) and \( w(r) = r^2. \)

(ii) By considering a trial function \( \psi_{\text{trial}}(r) = 1 - (r/a)^2, \) calculate an approximation to the lowest eigenvalue \( k_0^2. \)

(iii) Using the substitution \( \psi(r) = u(r)/r, \) determine the spherically-symmetric solutions of \( (†) \) and show that the exact lowest eigenvalue is \( k_0^2 = \pi^2/a^2. \)
Comment on the relation of this value with the approximate eigenvalue determined in (ii).

END OF PAPER