UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB & II (General)

Tuesday, 26 May, 2015 9:00 am to 12:00 pm

MATHEMATICS (1)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, ***6A****).*

Answers must be tied up in ***separate*** *bundles, marked* ***A, B or C*** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A ***separate*** *green master cover sheet listing all the questions attempted* ***must*** *also be completed.*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Green master cover sheet
Script paper

SPECIAL REQUIREMENTS
Calculator - students are permitted to bring an approved calculator.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1C

Consider a toroidal body defined parametrically in the Cartesian coordinate system $\mathbf{x} = (x, y, z)$ as the region satisfying

$$\begin{aligned} x &= (1 + r\sin\alpha)\cos\beta\,, \\ y &= (1 + r\sin\alpha)\sin\beta\,, \\ z &= r\cos\alpha\,, \end{aligned}$$

with $0 \leqslant r \leqslant R$, $-\pi \leqslant \alpha < \pi$ and $0 \leqslant \beta < 2\pi$ for constant $0 < R < 1$.

(a) For a toroidal coordinate system (r, α, β), determine the Cartesian components of the vectors $\mathbf{h}_r$, $\mathbf{h}_\alpha$, $\mathbf{h}_\beta$ such that the Cartesian differential $d\mathbf{x}$ is given by

$$d\mathbf{x} = \mathbf{h}_r dr + \mathbf{h}_\alpha d\alpha + \mathbf{h}_\beta d\beta\,,$$

and hence establish whether or not the toroidal coordinate system is orthogonal. Determine the Jacobian for the coordinate transformation. [8]

(b) Suppose the toroidal body is immersed in a vector field $\mathbf{F}(\mathbf{x}) = \boldsymbol{\nabla}\Omega + \boldsymbol{\nabla}\times\mathbf{U}$, where Ω is a scalar field and $\mathbf{U}$ is a vector field. Consider the integral

$$I = \int_S \mathbf{F}\cdot d\mathbf{S}\,,$$

where S is the surface of the body and $d\mathbf{S}$ is an element of vector area. Why does I not depend on $\mathbf{U}$? [3]

(c) Determine I for the case $\Omega = z^4 - xyz + e^{-3y}\cos 3x + e^{2x}\sin 2y$. [9]

2C

In two spatial dimensions the time evolution of a scalar field $u(x, y, t)$ is given by

$$\frac{\partial u}{\partial t} = \nabla^2 u + \beta \frac{\partial u}{\partial x}, \qquad (\star)$$

for constant β.

(a) Consider the domain $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$ with boundary conditions $\partial u/\partial y = 0$ on $y = 0$ and $u = 0$ on the other three boundaries. Use separation of variables to determine the general solution of $(\star)$ satisfying the boundary conditions and show that at $t = 0$ this reduces to

$$u(x, y, 0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} e^{-\frac{\beta}{2}x} \sin(m\pi x) \cos\left(\tfrac{2n-1}{2}\pi y\right) ,$$

for some set of constants A_{mn}. [14]

(b) Determine the constants A_{mn} required for u to satisfy the initial condition

$$u(x, y, 0) = x(1-x) e^{-\frac{\beta}{2}x} \cos\left(\tfrac{3}{2}\pi y\right) .$$

[6]

3B

(a) Use the method of Green's functions to solve

$$\frac{d^2y}{dx^2} - y = f(x)$$

for $0 \leqslant x \leqslant 1$, with the boundary conditions $y(0) = y(1) = 0$. [9]

[*You may use the identity* $\sinh a \cosh b - \cosh a \sinh b = \sinh(a - b)$.]

(b) A forced damped harmonic oscillator satisfies the equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + \left(1 + \mu^2\right) x = g(t)\,,$$

where μ is a positive constant.

(i) Solve this equation for $t \geqslant 0$ with initial conditions $x = dx/dt = 0$ at $t = 0$. [7]

(ii) Assume that $|g(t)| < Ce^{-at}$ where C and a are constants with $a > 0$. Prove that $x(t) \to 0$ as $t \to \infty$. [4]

4C

A radio station wishes to analyse its broadcast of the signal $a(t)$ using Fourier analysis. As a test signal, the radio station chooses a single pulse given by

$$a(t) = \begin{cases} 1 & \text{if } |t| < 1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

(a) Determine $\tilde{a}(\omega)$, the Fourier transform of $a(t)$. [3]

(b) Due to bandwidth limitations, the radio station decides to filter the signal so that it broadcasts

$$b(t) = \int_{-\infty}^{\infty} a(s) f(t-s)\, ds\,,$$

where the filter $f(t)$ is defined as

$$f(t) = \begin{cases} 1-|t| & \text{if } |t| < 1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Determine $\tilde{f}(\omega)$, the Fourier transform of $f(t)$, and hence use the convolution theorem to derive an expression for $\tilde{b}(\omega)$, the Fourier transform of $b(t)$. [7]

(c) Due to reflections from a mountain range, the signal received by the listeners can be modelled by

$$r(t) = b(t) + \alpha b(t-\tau)\,,$$

where α is the relative strength of the reflected signal and τ is the delay in receiving the reflection. Determine $\tilde{r}(\omega)$, the Fourier transform of $r(t)$. [4]

(d) Measurements of the received signal suggest it is well approximated by $s(t)$ with Fourier transform

$$\tilde{s}(\omega) = 2\left(1 + \epsilon\omega^2\right) e^{-\omega^2/4}$$

for some constant ϵ. Determine $s(t)$. [6]

[*You may assume* $\int_{-\infty}^{\infty} e^{-(z-ia)^2} dz = \sqrt{\pi}$ *for real* a.]

5A

(a) Let A and B be $n \times n$ Hermitian matrices that commute, i.e., $\mathsf{AB} = \mathsf{BA}$. Assuming that the eigenvalues of A and B are non-degenerate, show that the eigenvectors of A and B are the same so that A and B may be written as $\mathsf{A} = \mathsf{U}\Lambda_{\mathsf{A}}\mathsf{U}^\dagger$ and $\mathsf{B} = \mathsf{U}\Lambda_{\mathsf{B}}\mathsf{U}^\dagger$, where Λ_{A} and Λ_{B} are diagonal matrices and U is unitary. [4]

For such matrices A and B, using this result, or otherwise, show that

$$\exp(\mathsf{A})\exp(\mathsf{B}) = \exp(\mathsf{A} + \mathsf{B}),$$

where the exponential of a square matrix A is defined by the series

$$\exp(\mathsf{A}) = \mathsf{I} + \mathsf{A} + \frac{1}{2!}\mathsf{A}^2 + \frac{1}{3!}\mathsf{A}^3 + \cdots,$$

with I the identity matrix. [4]

(b) For general $n \times n$ matrices X and Y, verify that

$$\exp(\epsilon\mathsf{X})\exp(\epsilon\mathsf{Y}) = \exp\left(\epsilon\mathsf{X} + \epsilon\mathsf{Y} + \tfrac{1}{2}\epsilon^2[\mathsf{X},\mathsf{Y}]\right) + O\left(\epsilon^3\right),$$

where ϵ is an arbitrary parameter and

$$[\mathsf{X},\mathsf{Y}] \equiv \mathsf{XY} - \mathsf{YX}.$$

[5]

(c) For the matrix

$$\mathsf{M} = \begin{pmatrix} 0 & a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{pmatrix},$$

where a and b are real, show that

$$\exp(\mathsf{M}) = \begin{pmatrix} \cosh a & \sinh a & 0 & 0 \\ \sinh a & \cosh a & 0 & 0 \\ 0 & 0 & \cos b & \sin b \\ 0 & 0 & -\sin b & \cos b \end{pmatrix}.$$

[7]

6A

(a) Explain how to diagonalize a real symmetric matrix A. [4]

(b) Describe the quadratic surface Σ in $\mathbb{R}^3$ defined by

$$5x_1^2 - 8x_1x_2 + 5x_2^2 + 9x_3^2 = 9\,,$$

specifying the principal axes and, where appropriate, the semi-axis lengths. [6]

Show that Σ intersects the surface defined by

$$x_1^2 + x_2^2 + x_3^2 = 4$$

in a pair of circles, and find their orientations, radii, and centres. [5]

(c) On a general quadratic surface defined by $\mathsf{x}^T\mathsf{A}\mathsf{x} = 1$, with A a real symmetric matrix, show that the squared distance from the origin, $\mathsf{x}^T\mathsf{x}$, is extremised for x an eigenvector of A. [5]

7B

(a) Derive the Cauchy–Riemann equations for the analytic function

$$f(z) = u(x,y) + iv(x,y)\,,$$

where $z = x + iy$. [2]

(b) Determine the analytic function $f(z)$ if $u(x,y) = x\cos x\cosh y + y\sin x\sinh y$. [7]

(c) Assume that $g(z)$ is analytic and $|g(z)|$ is constant. Prove that $g(z)$ is constant. [6]

(d) Calculate the Taylor series of the function

$$h(z) = \frac{2z}{z^2+1}$$

about $z = 1$ and state its radius of convergence. [5]

[*Hint: use partial fractions.*]

8B

Consider the equation

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+n^2y=0\,. \qquad (\star)$$

(a) Show that $x=0$ is an ordinary point, and determine the nature of the points $x=\pm1$ and $x=\infty$. [6]

(b) Explain how to construct two independent solutions of $(\star)$ as power series about $x=0$. What is the recurrence relation for the coefficients of these series? [6]

(c) Use the ratio test to determine the radius of convergence of each series. How are these related to the location of the singular points of $(\star)$? [3]

(d) Show that polynomial solutions of $(\star)$ exist when n is an integer. With the condition $y(1)=1$, determine these solutions for the cases $n=0$, 1, 2, and 3. [5]

9A

(a) State the Euler–Lagrange equation that determines the extrema of the functional $F[z]$ of the function $z(x)$, where

$$F[z] = \int_{\alpha}^{\beta} f(z, z'; x)\, dx\,,$$

with primes denoting differentiation with respect to x. [2]

If f does not depend explicitly on x, show that

$$f - z' \frac{\partial f}{\partial z'} = \text{const.}$$

when $z(x)$ extremises $F[z]$. [4]

Explain how to determine the extrema of F subject to the constraint that a further functional $G[z]$ is constant. [2]

(b) An inextensible string of total length $\pi a/2$ hangs under its own weight in the x–z plane, with its endpoints fixed at $z = 0$ and $x = \pm a/\sqrt{2}$. The mass per unit length of the string, μ, is uniform. The gravitational potential $\Phi(z)$ varies with z, and is defined such that the potential energy of an element of the string of mass δm is $\delta m \Phi$. Parameterising the path of the string as $z(x)$, show that the total gravitational potential energy is

$$V[z] = \mu \int_{-a/\sqrt{2}}^{a/\sqrt{2}} \Phi(z) \sqrt{1 + z'^2}\, dx\,.$$

[2]

The shape adopted by the string is such as to minimise V subject to the constraint of a fixed length. Show that $z(x)$ satisfies

$$\frac{\mu\Phi(z) - \lambda}{\sqrt{1 + z'^2}} = \text{const.}\,,$$

where λ is a constant. [5]

Determine a suitable $\Phi(z)$ if the string is to hang along an arc of a circle of radius a. [5]

10A

(a) Consider the functionals

$$F[y] = \int_{\alpha}^{\beta} \left[p(x) \left(y'\right)^2 + q(x) y^2 \right] dx\,, \qquad G[y] = \int_{\alpha}^{\beta} w(x) y^2 \, dx\,,$$

where $p(x) > 0$, $q(x) \geqslant 0$, and $w(x) > 0$ for $\alpha < x < \beta$, and primes denote differentiation with respect to x. Show that if suitable boundary conditions are imposed at $x = \alpha$ and $x = \beta$, the ratio $F[y]/G[y]$ is extremised when y satisfies the Sturm–Liouville eigenvalue equation

$$-\left[p(x)y'\right]' + q(x)y = \lambda w(x) y\,. \qquad (\star)$$

How do the eigenvalues λ relate to the extremal values of $F[y]/G[y]$? [4]

Hence explain the Rayleigh–Ritz method for estimating the lowest eigenvalue of $(\star)$. [3]

(b) Pressure waves in a spherical cavity of radius a satisfy

$$\nabla^2 \psi + k^2 \psi = 0\,, \qquad (\dagger)$$

with k a real constant. The function ψ is bounded everywhere and vanishes on $r = a$.

(i) For spherically-symmetric solutions $\psi(r)$, show that $(\dagger)$ reduces to an ordinary differential equation that can be written in Sturm–Liouville form with $p(r) = r^2$, $q(r) = 0$, and $w(r) = r^2$. [3]

(ii) By considering a trial function $\psi_{\text{trial}}(r) = 1 - (r/a)^2$, calculate an approximation to the lowest eigenvalue k_0^2. [5]

(iii) Using the substitution $\psi(r) = u(r)/r$, determine the spherically-symmetric solutions of $(\dagger)$ and show that the exact lowest eigenvalue is $k_0^2 = \pi^2/a^2$.

Comment on the relation of this value with the approximate eigenvalue determined in (ii). [5]

END OF PAPER