

NST1

NATURAL SCIENCES TRIPOS

Part IB

Tuesday 16 June 2026 9:00am to 12:00pm

MATHEMATICS (2)**Read these instructions carefully before you begin:**

You may submit answers to no more than **six** questions. All questions carry the same number of marks.

The approximate number of marks allocated to a part of a question is indicated in the right-hand margin.

Write on **one** side of the paper only and begin each answer on a separate sheet.

At the end of the examination:

Each question has a number and a letter (for example, **8B**).

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

A **separate** green main cover sheet listing all the questions attempted **must** also be completed.

Every cover sheet must bear your examination number and desk number.

Tie up your answers and cover sheets into a **single bundle**, with the main cover sheet on top, and then the cover sheet and answer for each question, in the numerical order of the questions.

Calculators and other electronic or communication devices are not permitted in this examination.

STATIONERY REQUIREMENTS

6 gold cover sheets
Green main cover sheet
Script paper
Rough paper
Treasury tag

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1C

Consider the eigenvalue problem

$$\frac{d^2y}{dx^2} + \frac{2}{(x+1)} \frac{dy}{dx} + \frac{\lambda}{(x+1)^2} y = 0, \quad \text{in } 0 < x < 1, \quad \text{with } y(0) = y(1) = 0. \quad (*)$$

(a) Write problem (*) in Sturm-Liouville form. [4]

(b) Find the eigenvalues and their corresponding orthonormal eigenfunctions for problem (*). In particular, observe that $\lambda = 0$ is not an eigenvalue for problem (*). [8]

(c) Consider the boundary-value problem

$$\frac{d^2y}{dx^2} + \frac{2}{(x+1)} \frac{dy}{dx} = F(x), \quad \text{in } 0 < x < 1, \quad \text{with } y(0) = y(1) = 0, \quad (**)$$

where $F(x)$ be a given function on $[0, 1]$.

(i) Let $F(x) = (1+x)^{-5/2}$. Find the solution to the inhomogeneous boundary-value problem (**) as an eigenfunctions expansion. [6]

(ii) Let $F(x) = 0$. Use part (b), or otherwise, to prove that the only solution to the boundary-value problem (**) is $y(x) = 0$. [2]

2A

The gravitational potential outside the Earth may be approximated by a solution of Laplace's equation. In the absence of external influences, the Earth is approximately spherical with radius a . Due to tidal effects caused by a distant body, the surface of the Earth is slightly deformed. In spherical coordinates, the surface is described by

$$r = a + \varepsilon P_2(\cos \theta), \quad 0 < \varepsilon \ll a,$$

where

$$P_2(\cos \theta) = \frac{1}{2}(3 \cos^2 \theta - 1)$$

and (r, θ, ϕ) are standard spherical polar coordinates.

Assume that:

- the gravitational potential $\Phi(r, \theta)$ is independent of the azimuthal angle ϕ ,
 - Φ satisfies Laplace's equation in the region exterior to the Earth,
 - the surface of the Earth is an equipotential
 - the potential far from earth satisfies $\Phi(r) \sim -\frac{GM}{r}$ as $r \rightarrow \infty$, where G and M are constants.
- (a) Using separation of variables, find the most general axisymmetric solution of Laplace's equation outside the Earth which is regular as $r \rightarrow \infty$.
[Hint: In spherical polar coordinates (r, θ, ϕ) , the axisymmetric Laplace equation acts on a function $f(r, \theta)$ as:

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) = 0$$

You may assume that solutions to Legendre's differential equation, $\frac{d}{dx} \left[(1-x^2) \frac{dy}{dx} \right] + n(n+1)y = 0$, that are regular at $x = \pm 1$ are the Legendre polynomials $P_n(x)$ for integer $n \geq 0$.]

[6]

- (b) By expanding the boundary condition to first order in ε , determine the gravitational potential outside the earth to first order in ε .

[14]

3C

For a domain V in $\mathbb{R}^3$ with boundary S , we seek a solution u to Poisson's equation with Dirichlet boundary condition on S :

$$\begin{cases} \nabla^2 u(\mathbf{r}) &= \rho(\mathbf{r}) & \text{in } V, \\ u(\mathbf{r}) &= f(\mathbf{r}) & \text{on } S. \end{cases} \quad (*)$$

- (a) State the definition of Green's function $G(\mathbf{r}, \mathbf{r}')$ for problem $(*)$. Use Green's identity to express the solution u to problem $(*)$ as an integral in terms of $G(\mathbf{r}, \mathbf{r}')$, ρ and f .

[5]

- (b) (i) Write the explicit form of $G_0(\mathbf{r}, \mathbf{r}')$, the fundamental solution of Poisson's equation in $\mathbb{R}^3$.

[1]

- (ii) Let $V_1 = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$. Find an explicit form of the Green's function $G_1(\mathbf{r}, \mathbf{r}')$ associated with problem $(*)$ in the domain V_1 . Assuming $f(\mathbf{r}) \neq 0$, use part (a) to express the solution u to problem $(*)$ in the domain V_1 as an integral in terms of ρ , f and the explicit formulas for $G_1(\mathbf{r}, \mathbf{r}')$ and its derivatives.

[8]

- (iii) Let $V_2 = \{(x, y, z) \in \mathbb{R}^3 : x > 0, z > 0\}$. Find the Green's function $G_2(\mathbf{r}, \mathbf{r}')$ associated with problem $(*)$ in the domain V_2 .

[6]

4C

- (a) Use contour integration to compute the following integral

$$I = \int_0^\infty \frac{\cos x}{(x^2 + 1)^2} dx.$$

[10]

- (b) Let n be a positive integer. Use contour integration to show that

$$J = \int_0^\pi (\sin \theta)^{2n} d\theta = \frac{(2n)!}{2^{2n}(n!)^2} \pi.$$

[10]

5A

In this problem, you should assume that the Fourier transform of a function $f(t)$ is given by

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} dt f(t) e^{-i\omega t}.$$

Consider a particle at position $x(t)$ experiencing a force $F(t)$. The position is determined by the following differential equation:

$$\ddot{x} + 2\dot{x} + 5x = F(t),$$

where dots indicate derivatives with respect to time. For $t \leq 0$, $x = \dot{x} = 0$. Assume that $F(t) = 0$ for $t < 0$ and $F(t) = F_0 e^{-\epsilon t}$ for $t \geq 0$, where F_0, ϵ are positive constants.

a) Using Fourier transform methods, show that the solution can be written as follows:

$$\tilde{x}(\omega) = \tilde{F}(\omega) R(\omega).$$

where $R(\omega)$ is a transfer function, which you should specify,

[5]

b) Hence, by taking the inverse Fourier Transform, find the displacement of the particle as a function of time $x(t)$ for times $t > 0$.

[10]

c) Now assume $\epsilon = 1$. At what time does $x(t)$ reach its maximum value, and what is this maximum?

[5]

6A

A Cartesian tensor is said to be *isotropic* if its components are unchanged under all proper orthogonal transformations (pure rotations) of the coordinate axes.

For any linear transformation, you may assume that vectors transform as

$$v'_i = L_{ij}v_j,$$

second-order tensors transform as

$$T'_{ij} = L_{ip}L_{jq}T_{pq},$$

and the Levi-Civita symbol satisfies

$$\varepsilon'_{ijk} = (\det L) L_{ip}L_{jq}L_{kr}\varepsilon_{pqr},$$

where L_{ij} is a transformation matrix. (Repeated indices are summed.)

(a) Show that the only isotropic vector is the zero vector. [2]

(b) Show that any isotropic second-order tensor must be of the form

$$T_{ij} = \lambda\delta_{ij},$$

for some scalar λ . [4]

(c) Let A_{ij} be a constant antisymmetric second-order tensor and define

$$v_i = \varepsilon_{ijk}A_{jk}.$$

Show that v_i transforms as a vector under proper orthogonal transformations. [4]

(d) Let

$$K_{ij} = \int_V x_i x_j r^2 e^{-(r^2)} dV,$$

where V is all space and r is the radial coordinate $r^2 = \sum_i x_i^2$. Evaluate K_{ij} . [6]

(e) Let

$$M_{ij} = \int_V x_i x_j e^{-x_1^2/a^2 - x_2^2/b^2 - x_3^2/c^2} dV,$$

where V is all space and a, b, c are constants. Evaluate M_{ij} . [4]

7A

A rigid uniform disk of mass m and radius R lies on a frictionless horizontal table and is free to move in the plane and to rotate about its centre. The position of the disk's centre of mass can be described with cartesian coordinates (x, y) .

Two identical ideal springs (spring constant k , rest length l_0) are attached to the disk at two diametrically opposite points on its rim. The orientation of the disk is described by an angle θ measured from the x -axis to the line joining these two attachment points. The other ends of the springs are fixed to two walls so that, in equilibrium (with $\theta = 0$), both springs lie along the same straight line (the x -axis) and each has equilibrium length $l > l_0$ (i.e. the springs are pre-tensioned).

- a) Write the Lagrangian for small in-plane motions (x, y, θ) about equilibrium, keeping terms up to second order. [*Hint: you may assume that the moment of inertia for a uniform disk is $I = \frac{1}{2}mR^2$ and that the rotational kinetic energy is given by $\frac{1}{2}I \left(\frac{d\theta}{dt}\right)^2$.]* [13]
- b) Determine the normal modes and their angular frequencies. [7]

8B

- (a) Let H and K be two distinct subgroups of a group G . Prove that $H \cup K$ is a group if and only if either $H \subset K$ or $K \subset H$. [4]
- (b) What does it mean for a subgroup $H \subset G$ to be *normal*? [1]
- (c) Consider D_4 , the group of symmetries of a square. How many elements does D_4 have? Carefully explain the geometrical action of each element on the square. Construct the multiplication table for D_4 . Is this group Abelian? Justify your answer. [9]
- (d) How many subgroups of order 2 does D_4 have? List them explicitly. For each such subgroup, determine whether it is a normal subgroup of D_4 , giving a brief justification in each case. Explain whether the union of any two distinct, order 2 subgroups of D_4 is a group. [6]

9B

- (a) What does it mean for a finite group G to be *generated* by elements $g_1, g_2, \dots, g_k$?
What does it mean for an element $g \in G$ to have *order* n ? [2]

- (b) The *centre* $Z(G)$ of G is defined by

$$Z(G) = \{g \in G : gh = hg \text{ for all } h \in G\}.$$

Show that $Z(G)$ is a subgroup of G . [3]

Now let G be the group generated by elements a, b satisfying

$$a^4 = e, \quad b^2 = e, \quad bab^{-1} = a^{-1},$$

where e is the identity element, and a has order 4.

- (c) Show that every element of G is of the form $a^m b^n$ for $m \in \{0, 1, 2, 3\}$ and $n \in \{0, 1\}$.
Hence show that $|G| = 8$ and that G is non-abelian. [6]
- (d) Compute $Z(G)$ in this case. [4]
- (e) Classify all 1-dimensional complex representations of G . Hence determine the dimensions and number of irreducible complex representations of G . [*You may assume without proof that G has 5 conjugacy classes.*] [5]

10B

- (a) Define a *matrix representation* R of a group G . Show that if R is a matrix representation, then

$$R(g^{-1}) = (R(g))^{-1}$$

for all $g \in G$.

[5]

- (b) A group G has elements $\{e, a, b, c\}$ with multiplication rules

$$a^2 = b^2 = c^2 = e, \quad ab = c,$$

where e is the identity element. Show that G is Abelian and write down its multiplication table. Determine the conjugacy classes of G and hence deduce the number and dimensions of the irreducible representations of G . Construct the character table of G .

[8]

- (c) This group possesses a representation D_p given by

$$\begin{aligned} D_p(e) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & D_p(a) &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \\ D_p(b) &= \begin{pmatrix} 1 & p \\ 0 & -1 \end{pmatrix}, & D_p(c) &= \begin{pmatrix} -1 & -p \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

where $p \in \mathbb{R}$. Compute the character of D_p . Find a non-trivial subspace $U \subset \mathbb{R}^2$ that is invariant under D_p and hence decompose D_p into a sum of irreducible representations.

[7]

END OF PAPER