

NST1

NATURAL SCIENCES TRIPOS

Part IB

Monday 1 June 2026 2:00pm to 5:00pm

MATHEMATICS (1)**Read these instructions carefully before you begin:**

You may submit answers to no more than **six** questions. All questions carry the same number of marks.

The approximate number of marks allocated to a part of a question is indicated in the right-hand margin.

Write on **one** side of the paper only and begin each answer on a separate sheet.

At the end of the examination:

Each question has a number and a letter (for example, **8B**).

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

A **separate** green main cover sheet listing all the questions attempted **must** also be completed.

Every cover sheet must bear your examination number and desk number.

Tie up your answers and cover sheets into a **single bundle**, with the main cover sheet on top, and then the cover sheet and answer for each question, in the numerical order of the questions.

Calculators and other electronic or communication devices are not permitted in this examination.

STATIONERY REQUIREMENTS

6 gold cover sheets
Green main cover sheet
Script paper
Rough paper
Treasury tag

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1A

a) State the Divergence Theorem and Stokes' Theorem in $\mathbb{R}^3$. [2]

b) Show that $\nabla^2(\mathbf{U} \cdot \mathbf{V}) = (\nabla^2 \mathbf{U}) \cdot \mathbf{V} + \mathbf{U} \cdot (\nabla^2 \mathbf{V}) + 2 \sum_k (\nabla U_k) \cdot (\nabla V_k)$ where $\mathbf{U}$ and $\mathbf{V}$ are two vector fields. [6]

c) Consider the vector field $\mathbf{F} = (yz, xz, x^2)$ where x, y, z are Cartesian coordinates.

i) Using Stokes' Theorem, evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{l}$, where C is the unit circle $x^2 + y^2 = 1$ in the xy -plane, oriented counterclockwise. [6]

ii) Let S_1 be the part of the paraboloid $z = 1 - x^2 - y^2$ which is above the xy -plane. Let S_2 be the unit disk in the xy -plane (i.e., $x^2 + y^2 \leq 1$, $z = 0$). Use the divergence theorem to find the flux $\int_{S_1} \mathbf{F} \cdot d\mathbf{S}$ upward through S_1 . [6]

2A

a) Consider the heat equation on a rectangular domain $0 < x < L$, $0 < y < W$:

$$\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

with boundary conditions $T(0, y, t) = T_H$, $T(L, y, t) = T_C$ and $\frac{\partial T}{\partial y}(x, 0, t) = \frac{\partial T}{\partial y}(x, W, t) = 0$. Here α , T_H and T_C are positive constants and $T_H > T_C$. The initial condition is $T(x, y, 0) = T_0$ (constant).

Using separation of variables, find the solution $T(x, y, t)$ for all $t \geq 0$. Express your answer as an infinite series. [13]

b) The domain is now extended in the x direction so that $0 < x < L + L_2$, with two regions:

$$\frac{\partial T}{\partial t} = \begin{cases} \alpha \nabla^2 T & \text{for } 0 < x < L \\ \beta \nabla^2 T & \text{for } L < x < L + L_2 \end{cases}$$

Here α and β are positive constants. The boundary conditions are $T(0, y, t) = T_H$ and $T(L + L_2, y, t) = T_C$, with the top and bottom still insulated (i.e. $\frac{\partial T}{\partial y}(x, 0, t) = \frac{\partial T}{\partial y}(x, W, t) = 0$). At the interface $x = L$, the solution satisfies

$$T(L^-, y, t) = T(L^+, y, t), \quad k_1 \frac{\partial T}{\partial x} \Big|_{x=L^-} = k_2 \frac{\partial T}{\partial x} \Big|_{x=L^+}$$

where k_1 and k_2 are positive constants associated with each region.

Find the steady-state solution $T_s(x)$. How does the ratio k_1/k_2 affect the solution? [7]

3C

- (i) Construct the Green's function associated to the following boundary-value problem

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = f(x), \quad \text{in the infinite interval } 0 < x < \infty, \quad (*)$$

subject to boundary conditions $y(0) = 0$ and $\lim_{x \rightarrow \infty} y(x) = 0$.

[12]

- (ii) Use the Green's function constructed in part (i) to compute the exact solution to the boundary-value problem (*) for the case $f(x) = e^{-2x}$.

[8]

4A

The Fourier Transform of a function $f(x)$ is defined as:

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx} dx.$$

a) State and prove the convolution theorem for the Fourier Transform of the convolution $(h * g)(x) = \int_{-\infty}^{\infty} h(\tau)g(x - \tau)d\tau$ of two functions $h(x)$ and $g(x)$. [5]

b) Let $f(x)$ be a probability density function for a random variable X , defined for $-\infty < x < \infty$. Consider the “Top Hat” density function given by:

$$f(x) = \begin{cases} 1 & |x| \leq 1/2 \\ 0 & |x| > 1/2 \end{cases}.$$

Show that its Fourier Transform is $\tilde{f}(k) = \frac{\sin(k/2)}{k/2}$. [4]

c) Let $X_1, X_2, \dots, X_n$ be independent, identically distributed random variables, each with the density function $f(x)$ given in part (b). Assume that the density function of their sum, $S_n = \sum_{i=1}^n X_i$, is given by the n -fold convolution:

$$f_n(x) = (f * f * \dots * f)(x).$$

From the convolution theorem, derive an expression for the Fourier Transform of repeated convolutions. Hence write down an expression for the Fourier Transform $\tilde{f}_n(k)$ in terms of $\tilde{f}(k)$. [4]

d) To examine the behavior of the sum for large n , consider the scaled random variable $Z_n = \frac{1}{\sqrt{n}}S_n$. Argue that the Fourier Transform of the density $g_n(z)$ of Z_n is given by:

$$\tilde{g}_n(k) = \left[\tilde{f}\left(\frac{k}{\sqrt{n}}\right) \right]^n.$$

By using a Taylor expansion of $\tilde{f}\left(\frac{k}{\sqrt{n}}\right)$ for small arguments, or otherwise, show that in the limit $n \rightarrow \infty$:

$$\tilde{g}_n(k) \rightarrow e^{-k^2/C},$$

where you should specify the constant C . Invert this transform to find the limiting density $g(z)$.

[Hints: You may wish to recall that for a scaled random variable $Z = X/a$, the probability density is $g(z) = af(az)$, where g and f are the probability density functions for Z and X , respectively, and a is a constant; you may also assume the standard integral $\int_{-\infty}^{\infty} e^{-cx^2} e^{-ikx} dx = \sqrt{\frac{\pi}{c}} e^{-k^2/4c}$ for a constant $c > 0$.] [7]

5B

Let M be an invertible $n \times n$ complex matrix, with Hermitian conjugate $M^\dagger$ and inverse M^{-1} . Show that $(M^\dagger)^{-1} = (M^{-1})^\dagger$. [3]

What does it mean for a matrix to be *unitary*? If U is an $n \times n$ unitary matrix and I is the $n \times n$ identity matrix, show that $|\det(I + U)| \leq 2^n$. [4]

Let U be an $n \times n$ unitary matrix and suppose that $(I + U)$ is invertible. Show that $(I + U^\dagger)$ is invertible and that

$$A = (I + U)^{-1}(I - U)$$

obeys $A^\dagger = -A$. [8]

Hence (or otherwise) deduce a bound on $|\det(I + A)|$, and give an example of an $n \times n$ unitary U for which this bound is saturated. [5]

6B

Prove that every real symmetric matrix S is diagonalisable. [5]

Let $\Sigma_\kappa \subset \mathbb{R}^3$ be the surface

$$\frac{\kappa}{2}(x + y)^2 + 2xz - 2yz + z^2 = -4,$$

Construct a suitable matrix S such that $\Sigma_\kappa = \{\mathbf{x}^\top S \mathbf{x} = -4\}$, where $\mathbf{x} = (x, y, z)^\top$. What is the significance of choosing S to be symmetric? [4]

Find the shortest distance between Σ_κ and the origin for all real values of the parameter κ . In the case $\kappa = 0$, give the coordinates $\mathbf{x}$ of the point(s) on Σ_κ where this minimum distance is achieved. [9]

Sketch Σ_κ in the case $\kappa = -1$. [2]

7B

- (a) State Taylor's theorem, including a remainder term $R_N(z)$, for a function analytic in a disc $|z - z_0| < r$. Explain what it means to write

$$R_N(z) = O((z - z_0)^{N+1}) \quad (z \rightarrow z_0),$$

and what such a bound implies about the size of the truncation error near z_0 . [4]

- (b) Let $\text{Log}(z)$ denote the principal branch of the complex logarithm on $\mathbb{C} \setminus (-\infty, 0]$. Show that

$$-\text{Log}(1 - z) = z + \frac{1}{2}z^2 + O(z^3)$$

as $z \rightarrow 0$. Classify the type of singularity of $-\text{Log}(1 - z)$ at $z = 1$, justifying your answer carefully. [4]

- (c) Let $F(z)$ be the function

$$F(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}.$$

Determine the radius of convergence r of F . Explain why $F'(z)$ is given by term-by-term differentiation within this radius, and hence identify $F(z)$ within $|z| < r$. [8]

- (d) Now let $G(z)$ be the function

$$G(z) = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) z^n, \quad |z| < 1.$$

Find the limiting value of $G(z)$ as $z \rightarrow 1$ from below along the real axis. [4]

8C

Consider the second order differential equation for a function $y(x)$

$$x \frac{d^2 y}{dx^2} + \frac{dy}{dx} - y = 0. \quad (*)$$

- (i) Identify and classify all the singular points of equation $(*)$ at finite values of x . [4]
 (ii) Equation $(*)$ admits solutions of the form

$$y(x) = x^\sigma \sum_{n=0}^{\infty} a_n x^n, \quad (a_0 \neq 0).$$

Determine the possible values of σ and find the corresponding recursion relations satisfied by the coefficients a_n .

Using these recursion relations, verify that, for a suitable choice of a_0 , the solution that is regular at $x = 0$ is

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}. \quad (**)$$

[10]

- (iii) Show that any solution to equation $(*)$ that is independent of the solution $(**)$ must be singular at $x = 0$.

[6]

9C

- (a) A designer needs to determine the shape $y(x) \geq 0$ of a barrier to be built between the points $(x, y) = (-1, 0)$ and $(1, 0)$ with specified area $A > 0$ bounded between the barrier and the x -axis, that is

$$\int_{-1}^1 y(x) \, dx = A.$$

The designer has been told to minimise the cost function

$$C[y] = \int_{-1}^1 ((y(x))^2 + (y'(x))^2) \, dx,$$

subject to the constraints. Determine the optimal shape $y(x)$. [10]

- (b) Consider a light ray propagating in a medium in the (x, y) plane with refractive index μ .

(i) Explain what is meant by Fermat's Principle. [2]

- (ii) Assume the medium is the upper half plane $y > 0$ with refractive index $\mu(y) = \frac{1}{y}$. A light ray is fired from the point $(x, y) = (-1, 1)$ and is picked up by a receiver at the point $(x, y) = (1, 1)$. Calculate the path that the light follows. [8]

10C

Let $w : [0, 1] \rightarrow \mathbb{R}$ be a smooth function, with $w(0) = 0$ and $w(1) = 1$. Consider the functional

$$J[w] = \int_0^1 \left[e^x \left(\frac{dw}{dx} \right)^2 + (1+x)(w)^2 \right] dx.$$

- (a) Show that every smooth minimiser of J is a solution to the boundary-value problem

$$-\frac{d}{dx} \left(e^x \frac{dw}{dx} \right) + (1+x)w = 0 \quad \text{for } 0 < x < 1, \quad (*)$$

with boundary conditions $w(0) = 0$, $w(1) = 1$. [4]

- (b) We now seek an *approximate* solution of this boundary-value problem by constructing approximate minimisers of the functional J , as follows:

- (i) First, let v_0 be an arbitrary smooth function that satisfies the boundary conditions $v_0(0) = 0$ and $v_0(1) = 1$, but does not necessarily obey equation (*). Next, take a non-zero smooth function v_1 which has homogeneous boundary conditions, $v_1(0) = v_1(1) = 0$. We consider the one-parameter family of trial functions

$$w_1(x) = v_0(x) + a_1 v_1(x),$$

and choose the real parameter a_1 to minimise $J[w_1]$. Observe that w_1 satisfies the required boundary conditions. Write the equation that determines the value of the parameter a_1 which minimises $J[w_1]$ in terms of integrals involving v_0, v_1 and their derivatives. [6]

- (ii) Consider v_0 and v_1 as in (i) and introduce another smooth function v_2 , linearly independent of v_1 , which also satisfies $v_2(0) = v_2(1) = 0$. We consider the two-parameter family of trial functions

$$w_2(x) = v_0(x) + b_1 v_1(x) + b_2 v_2(x),$$

and choose the real parameters (b_1, b_2) to minimise $J[w_2]$. Observe again that w_2 satisfies the same boundary conditions as w . Write a system of equations that determines the parameters b_1, b_2 which minimise the value of $J[w_2]$ in terms of integrals involving v_0, v_1, v_2 and their derivatives. Moreover, explain briefly why the minimum value of $J[w_2]$ is less than or equal to the minimum value of $J[w_1]$. [10]

END OF PAPER