

NST0

NATURAL SCIENCES TRIPOS

Part IA

Wednesday 17 June 2026 9:00am to 12:00pm

MATHEMATICS (2)**Read these instructions carefully before you begin:**

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

You may submit answers to **all** of section A, and to no more than **five** questions from section B.

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on one side of the paper only and begin each answer on a separate sheet. (For this purpose, your section A attempts should be considered as one single answer.)

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Complete a gold cover sheet for your section A answer, and place it at the front of your answer to section A.

Complete a gold cover sheet for **each** section B answer, and place it at the front of your answer to that question.

A **separate** green main cover sheet listing all the questions attempted **must** also be completed. (Your section A answer should be recorded just as A: there is no need to list each individual short question.)

Every cover sheet must bear your blind grade number and desk number, and should NOT include your name or CRSid.

Tie up your answers and cover sheets into a **single bundle**, with the main cover sheet on top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS

6 gold cover sheets.

Green main cover sheet.

Script paper.

Treasury tag.

SPECIAL REQUIREMENTS

No calculators may be used.

No electronic devices may be used.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1X For the vectors $\mathbf{a} = (1, 0, 2)^\top$, $\mathbf{b} = (2, 0, -1)^\top$ and $\mathbf{c} = (1, 1, 1)^\top$, calculate:

(a) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$, [1]

(b) the angle between $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ and $\mathbf{b}$. [1]

2X Given that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$, evaluate $\int_0^{\infty} x^2 e^{-x^2} dx$. [2]

3X Consider the Taylor series expansion of $y = x \ln(x)$ about $x = 1$.

(a) Calculate the first non-zero term in this expansion. [1]

(b) Derive an expression for the n^{th} non-zero term in the expansion, where $n > 1$. [1]

4X Evaluate $\iint_D (x^2 + y^2) dx dy$, where D is the interior region of the unit circle centred at the origin. [2]

5X Find the eigenvalues of the matrix

$$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}. \quad [2]$$

6X For the scalar field $\phi(x, y) = (1 + x^2)y$, calculate:

(a) $\nabla \phi$, [1]

(b) $\nabla^2 \phi$. [1]

7X Find the Fourier series on $-\pi \leq x < \pi$ for the function $\cos^3 x$. [2]

8X Consider the probability density function,

$$p(r) = \begin{cases} Ar^2 & \text{if } 0 \leq r \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Calculate:

- (a) the value of A , [1]
- (b) the mean value of r . [1]

9X Consider the simultaneous equations

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & b \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

- (a) For what value of b do non-zero solutions exist? [1]
- (b) Give a geometrical description of these non-zero solutions. [1]

10X Calculate the vector area of the quadrilateral with vertices at points $A = (1, 0, 0)$, $B = (2, 2, 0)$, $C = (0, 1, 0)$, and $O = (0, 0, 0)$, travelling around the perimeter in the direction $A \rightarrow B \rightarrow C \rightarrow O \rightarrow A$. [2]

SECTION B

11T

- (a) Define the complex logarithm $\ln z$, for $z \in \mathbb{C}$ and $z \neq 0$, and state its relationship to the principal value of the logarithm. [1]
- (b) For each of the following equations, find all distinct solutions $z \in \mathbb{C}$, giving your answers in terms of r and θ , where $z = re^{i\theta}$.
- (i) $z = i^i$, [2]
- (ii) $z^3 = -1$, [2]
- (iii) $e^z = 1$, [2]
- (iv) $z^5 = (1 + e^{i\beta})$, where β is constant and $-\pi < \beta < \pi$. [3]
- (c) Consider the two curves in the Argand plane defined by $zz^* + z + z^* - 4 = 0$ and $zz^* - 5z - 5z^* + 8 = 0$, where z^* denotes the complex conjugate of z .
- (i) By completing the square, or otherwise, describe each curve geometrically. [2]
- (ii) Find the coordinates of any points of intersection of the two curves. [2]
- (iii) Sketch the curves in the same diagram in the Argand plane, showing any intersections. [2]
- (d) Let $f(z) = e^{ia}z + b$, where a and b are complex constants, $|e^{ia}| = 1$, and $z \in \mathbb{C}$ is variable.
- (i) Find the inverse function $w = f^{-1}(z)$. [2]
- (ii) Given that $f(z)$ satisfies $f\left(\frac{8(i-1)}{\sqrt{2}}\right) = -1$ and $f\left(\frac{7(i-1)}{\sqrt{2}}\right) = 0$, find all possible values of a and b . [2]

12V The continuous random variable X has the following cumulative distribution function:

$$P(X \leq x) = \begin{cases} 0 & x \leq a, \\ \frac{(x-a)^2}{N(c-a)} & a < x \leq c, \\ 1 - \frac{(b-x)^2}{N(b-c)} & c < x < b, \\ 1 & b \leq x, \end{cases}$$

where the parameters obey the inequality $a < c < b$, and N is a positive constant.

(a) Find the probability density function for X . [4]

(b) Find the normalising constant N . [3]

(c) Show that the expectation value of X , $E[X]$, is

$$E[X] = \frac{a + b + c}{p},$$

where p is an integer that you should determine. [5]

(d) Show that the variance of X , $\text{var}(X)$, is

$$\text{var}(X) = \frac{a^2 + b^2 + c^2 - ab - ac - bc}{q},$$

where q is an integer that you should determine. [5]

(e) Sketch the probability density function for X , clearly indicating the behaviour at $x = a$, $x = c$, and $x = b$. [3]

13Z

- (a) Consider the differential equation

$$x^2 \frac{d^2 y(x)}{dx^2} - x \frac{dy(x)}{dx} + y(x) = x,$$

defined for $x > 0$. By making the substitution $x = e^t$, or otherwise, find the general solution for $y(x)$. [8]

- (b) Solve the differential equation

$$\frac{d^2 y(x)}{dx^2} + 4y(x) = g(x)$$

subject to the boundary conditions $y(0) = 0$ and $\frac{dy}{dx}(0) = 1$, in each of the cases:

(i) $g(x) = 0$, [5]

(ii) $g(x) = 8 \sin^2 x$. [7]

14T (Matrix notation provides a convenient notation, but matrix methods are not essential to the solution of this question.)

- (a) Consider the change of variables from Cartesian coordinates (x, y) to plane polar coordinates (r, θ) , defined by $x = r \cos \theta$ and $y = r \sin \theta$, with $r > 0$ and $0 \leq \theta < 2\pi$.

- (i) By finding expressions for dx and dy in terms of r, θ, dr and $d\theta$, find a matrix

$$\mathbf{M}, \text{ with elements in terms of } r \text{ and } \theta, \text{ such that } \begin{pmatrix} dx \\ dy \end{pmatrix} = \mathbf{M} \begin{pmatrix} dr \\ r d\theta \end{pmatrix}. \quad [2]$$

- (ii) Let $f(r, \theta) = F(x, y)$. By using the chain rule to find expressions for $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$ in terms of $r, \theta, \frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$, find a matrix $\mathbf{N}$, with elements in terms of r and θ , such that

$$\begin{pmatrix} \frac{\partial f}{\partial r} \\ \frac{1}{r} \frac{\partial f}{\partial \theta} \end{pmatrix} = \mathbf{N} \begin{pmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial F}{\partial y} \end{pmatrix}. \quad [2]$$

- (iii) Hence, or otherwise, find matrices $\mathbf{J}$ and $\mathbf{K}$, with elements in terms of r and θ , such that

$$\begin{pmatrix} dr \\ r d\theta \end{pmatrix} = \mathbf{J} \begin{pmatrix} dx \\ dy \end{pmatrix}, \quad \begin{pmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial F}{\partial y} \end{pmatrix} = \mathbf{K} \begin{pmatrix} \frac{\partial f}{\partial r} \\ \frac{1}{r} \frac{\partial f}{\partial \theta} \end{pmatrix}. \quad [2]$$

- (iv) Let $d\mathbf{v} = \begin{pmatrix} dx \\ dy \end{pmatrix}$. Using $ds^2 = |d\mathbf{v}|^2 = (d\mathbf{v})^\top (d\mathbf{v})$, show that $ds^2 = dx^2 + dy^2$. [1]

- (v) Write ds^2 in terms of dr and $d\theta$, and comment briefly on the result. [2]

- (b) Consider $f(\phi, X) = F(x, y)$, where the change of coordinates is given by $x = X$ and $y = X \tan \phi$, with $X > 0$ and $0 \leq \phi < \frac{\pi}{2}$.

- (i) Find a matrix $\mathbf{Q}$, with elements in terms of x and y , such that

$$\begin{pmatrix} \frac{\partial f}{\partial \theta} \\ \frac{\partial f}{\partial X} \end{pmatrix} = \mathbf{Q} \begin{pmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial F}{\partial y} \end{pmatrix}. \quad [4]$$

- (ii) State and explain the relationship between $\frac{\partial f}{\partial X}$ and $\frac{\partial F}{\partial x}$. [4]

- (iii) Let $F(x, y) = \frac{(x-1)^2 y}{(x^2 + y^2)^{1/2}}$. Find $\frac{\partial f}{\partial X}$ in (x, y) coordinates. Find $\frac{\partial f}{\partial X}$ in (ϕ, X) coordinates. [3]

15V

- (a) Two solid objects are defined in cylindrical polar coordinates (r, θ, z) . Find the volume of each object.

(i)

$$0 \leq z \leq H, \quad 0 \leq \theta < 2\pi, \quad 0 \leq r \leq z \cos(\theta/4). \quad [5]$$

(ii)

$$0 \leq z \leq H, \quad \frac{z}{H} - \frac{\pi}{6} \leq \theta \leq \frac{z}{H} + \frac{\pi}{6}, \quad 0 \leq r \leq \exp\left(3 - \frac{z}{H}\right). \quad [5]$$

- (b) Evaluate the triple integral

$$J = \iiint_S |x| e^{-y} \, dx \, dy \, dz$$

in Cartesian coordinates, where S is defined by the inequalities: $-2 \leq x \leq 2$, $0 \leq y \leq |x|$, $-5 \leq z \leq xy$. [10]

16S Let C be the curve in the plane parameterised by

$$\mathbf{r}(t) = (x(t), y(t))^T = (t^2, t^3 - t)^T, \quad 0 \leq t \leq 1.$$

A continuous vector field $\mathbf{F}(x, y)$ is given by

$$\mathbf{F}(x, y) = (2xy + y^2, x^2 + 2xy)^T.$$

- (a) Find $d\mathbf{r}(t)/dt$, and hence express dx and dy in terms of dt . Sketch the curve C in the x - y plane, indicating its start ($t = 0$) and end ($t = 1$) points and the direction corresponding to increasing value of t . [2]

- (b) Compute the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

by reducing it to integration over the parameter t . [4]

- (c) Determine whether $\mathbf{F}$ is conservative on $\mathbb{R}^2$. If it is conservative, find a potential function $\phi(x, y)$ such that $\nabla\phi = \mathbf{F}$, and use it to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ again. [4]

- (d) A continuous vector field $\mathbf{G}(x, y)$ is given by

$$\mathbf{G}(x, y) = (-y, x)^T.$$

- (i) Show that $\mathbf{G}$ is not conservative on $\mathbb{R}^2$. [2]
 (ii) Evaluate the line integral $\int_C \mathbf{G} \cdot d\mathbf{r}$. [4]
 (iii) Let D be the straight line segment from $(0, 0)$ to $(1, 0)$. Compute $\int_D \mathbf{G} \cdot d\mathbf{r}$ and compare it with your answer for $\int_C \mathbf{G} \cdot d\mathbf{r}$. [2]
 Give a brief explanation of what this illustrates about conservative and non-conservative vector fields. [2]

17W The vector field $\mathbf{F}(\mathbf{r})$ is defined for $r > 0$, where $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$, by,

$$\mathbf{F}(\mathbf{r}) = \phi(r) \left(\frac{\mathbf{r}}{r} + \mathbf{a} \right), \quad \phi(r) = Ar^\alpha,$$

where A and α are real constants and $\mathbf{a}$ is a constant vector.

Evaluate:

(a) the divergence and curl of $\mathbf{F}$, [8]

(b) the Laplacian $\nabla^2 \phi(r)$, [5]

(c) the outward flux

$$\Phi = \iint_S \mathbf{F} \cdot d\mathbf{S}$$

through the surface S of the sphere of unit radius centred at the origin, [5]

(d) the surface integral

$$\mathbf{G} = \iint_S e^{\phi(r)} d\mathbf{S},$$

where S is the same sphere as defined in (c). [2]

18Z Consider a real $n \times n$ matrix $\mathbf{M}$ such that $\mathbf{M} = \mathbf{M}^\top$.

(a) What properties hold for the eigenvalues and eigenvectors of $\mathbf{M}$ which do not generally hold for square matrices? [2]

(b) Give expressions for the trace and determinant of $\mathbf{M}$ in terms of its eigenvalues. [2]

(c) The diagonalized form of $\mathbf{M}$ is $\mathbf{D}$, where the diagonal elements of $\mathbf{D}$ are the eigenvalues of $\mathbf{M}$. Write down matrices $\mathbf{P}$ and $\mathbf{Q}$ such that $\mathbf{D} = \mathbf{P}\mathbf{M}\mathbf{Q}$. [3]

(d) Consider a real $n \times n$ matrix $\mathbf{N}$ such that $\mathbf{N} = \mathbf{N}^\top$. The (all distinct) eigenvalues of $\mathbf{M}$ are $\{\lambda_i\}$ and the (all distinct) eigenvalues of $\mathbf{N}$ are $\{\mu_i\}$.

Show that $\mathbf{M}\mathbf{N} = \mathbf{N}\mathbf{M}$ if and only if every eigenvector of $\mathbf{M}$ is also an eigenvector of $\mathbf{N}$. In this case, determine whether the matrices $\mathbf{P}$ and $\mathbf{Q}$ found in (c) diagonalize $\mathbf{N}$. [7]

(e) Consider the matrices

$$\mathbf{M} = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}, \quad \mathbf{N} = \begin{pmatrix} -2 & 4 \\ 4 & -2 \end{pmatrix}.$$

Calculate $\mathbf{M}\mathbf{N} - \mathbf{N}\mathbf{M}$. Diagonalize $\mathbf{M}$ and $\mathbf{N}$, showing your working. [6]

19R*

- (a) Explain what is meant by the notation

$$f(x) = O(g(x)) \quad \text{as } x \rightarrow \infty.$$

Let $f(x)$ be differentiable for $x > 0$, and suppose that

$$f'(x) = O(x^{-3/2}) \quad \text{as } x \rightarrow \infty.$$

By considering $|f(x) - f(y)|$ for large x and y , or otherwise, show that for any $\epsilon > 0$ there are real constants A, X , with $X > 0$, such that $A < f(x) < A + \epsilon$ for all $x > X$. [6]

- (b) For which real values of α does the series

$$\sum_{n=1}^{\infty} \left[\sqrt{n^4 + \alpha^2 \sin^2(\alpha n)} - n^2 \right] \cos(\alpha n)$$

converge? Justify your answer. [7]

- (c) Define the sequence (b_n) for positive integer n by

$$b_n = \int_n^{n+1} \frac{\sin x}{x} dx.$$

- (i) Show that $b_n = O(n^{-1})$ as $n \rightarrow \infty$. [2]
(ii) By considering partial sums, or otherwise, determine whether the series

$$\sum_{n=1}^{\infty} b_n$$

converges. [5]

20Y*

(a) Consider the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

for real-valued functions $u(x, t)$ for $0 \leq x \leq 1$ and for $t \geq 0$. Find the solution that decays with increasing t subject to the boundary conditions

$$u(0, t) = u(1, t) = 0$$

and the initial condition

$$u(x, 0) = x - x^2. \quad [10]$$

(b) Consider the equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

for real-valued functions $u(x, t)$ for $0 \leq x \leq \pi$ and for $t \geq 0$. Find the solution subject to the boundary conditions

$$u(0, t) = u(\pi, t) = 0$$

with initial conditions

$$u(x, 0) = \sin x$$

and

$$\frac{\partial u(x, 0)}{\partial t} = 0. \quad [10]$$

END OF PAPER