

NST0

NATURAL SCIENCES TRIPOS

Part IA

Monday 15 June 2026 9:00am to 12:00pm

MATHEMATICS (1)**Read these instructions carefully before you begin:**

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

You may submit answers to **all** of section A, and to no more than **five** questions from section B.

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on one side of the paper only and begin each answer on a separate sheet. (For this purpose, your section A attempts should be considered as one single answer.)

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Complete a gold cover sheet for your section A answer, and place it at the front of your answer to section A.

Complete a gold cover sheet for **each** section B answer, and place it at the front of your answer to that question.

A **separate** green main cover sheet listing all the questions attempted **must** also be completed. (Your section A answer should be recorded just as A: there is no need to list each individual short question.)

Every cover sheet must bear your blind grade number and desk number, and should NOT include your name or CRSid.

Tie up your answers and cover sheets into a **single bundle**, with the main cover sheet on top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS

6 gold cover sheets.

Green main cover sheet.

Script paper.

Treasury tag.

SPECIAL REQUIREMENTS

No calculators may be used.

No electronic devices may be used.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A**1W** Complete the factorisation of

$$27^x + 8^x = (3^x + 2^x) \times (?) \quad [2]$$

into real factors.

2W Find all solutions for real x of the equation:

$$x - |x - 1| = 1. \quad [2]$$

3W Find:(a) the perimeter, and [1](b) the area [1]

of an equilateral triangle if the radius of the inscribed circle is equal to unity.

4W Let the vector

$$\mathbf{b} = (b_x, b_y)^\top = \sum_{n=0}^{100} (n, -n + 1)^\top.$$

Find:

(a) b_x , [1](b) $b_x + b_y$. [1]**5W** Evaluate the following integrals:(a)
$$\int \frac{2x}{x^2 + 1} \mathrm{d}x, \quad [1]$$
(b)
$$\int_{2/\pi}^{4/\pi} [\arctan(x) + \arctan(1/x)] \mathrm{d}x. \quad [1]$$

6W Differentiate with respect to x :

(a) $\arccos(-x) + \arccos(x),$ [1]

(b) $2^x + 2.$ [1]

7W Express

$$\frac{3x + 4}{x^2 + 3x + 2}$$
 [2]

as a sum of partial fractions.

8W Sketch the graphs of the following real functions:

(a) $x + \frac{1}{x},$ [1]

(b) $\sqrt{x+1} + 1.$ [1]

Coordinates of extrema (if any) are not required.

9W A random walker makes steps of unit length along the x -axis in the positive direction with probability p and in the negative direction with probability $1 - p$. The random walk starts from the origin. Find the probability of returning to the origin after:

(a) two steps, [1]

(b) five steps. [1]

10W What is the shortest distance between the curve

$$x^2 + y^2 - 6x + 4y + 12 = 0$$

and

(a) the x -axis, [1]

(b) the y -axis? [1]

SECTION B

11S (a) Let (x, y, z) be Cartesian coordinates in three-dimensional space. Consider the equation

$$2x - y + z = 0.$$

- (i) What geometrical object is described by this equation? What properties determine the object uniquely? [1]
- (ii) Find the two unit vectors normal to this object. [2]
- (iii) Find the vector equation of a line that lies entirely within this object, that passes through the point $(1, 1, -1)$, and that is parallel to the plane $z = 0$. [2]

(b) The points A , B and C have position vectors

$$\mathbf{a} = (1, 0, 1)^T, \quad \mathbf{b} = (2, 1, 0)^T, \quad \mathbf{c} = (0, 1, 2)^T.$$

- (i) Are the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ orthogonal to each other? [1]
- (ii) Find a vector normal to the plane through the points A , B , and C . [2]
- (iii) Hence find the explicit vector equation for all points $\mathbf{r}$ in this plane. [1]
- (iv) Compute the area of the triangle with vertices A , B , and C . [2]

(c) A line L has vector equation

$$\mathbf{r} = \mathbf{d} + \lambda(2, 1, -1)^T.$$

- (i) For $\mathbf{d} = (1, -1, 2)^T$, determine whether the line L intersects the geometrical object described by $x + z + 5 = 0$. If it does, find the point of intersection. [2]
- (ii) If instead $\mathbf{d} = (1, -2, 1)^T$, determine whether the line L is parallel to the line passing through $(1, 2, 3)$ and $(3, 2, 1)$. [2]

(d) A line L has vector equation

$$\mathbf{r} = \frac{\lambda}{\sqrt{2}}(1, 1, 0)^T.$$

- (i) Write an expression for the distance of a point P from the line L . [2]
- (ii) Hence derive an explicit equation for the points P on the surface of a cylinder with axis the line L and radius R . [3]

12R

- (a) State Taylor's theorem for a function $f(x)$ expanded about $x = a$, including the remainder term. [3]
- (b) (i) Calculate the first three terms in the Taylor series for $f(x) = \sqrt{4+x}$ about $x = 0$. [3]
- (ii) Determine whether $f'''(x)$ is monotonically increasing, decreasing, or neither for $x > 0$. Justify your answer. [2]
- (iii) Use these results to give an estimate for $f(1/8)$ in terms of positive and negative integer powers of 2, accurate to within an error of 2^{-18} , explaining carefully how you justify this bound on the error. [6]
- (c) Find the first three non-zero terms in the Maclaurin series for

$$f(x) = \sin\left(\sqrt{\frac{\pi}{9} + x}\right). \quad [6]$$

13X Calculate the following integrals:

(a)
$$\int \cos(a\sqrt{x}) \, dx, \quad a \in \mathbb{R}, \quad [6]$$

(b)
$$\int e^{2x} \sin x \, dx, \quad [4]$$

(c)
$$\int_0^1 \frac{2x^3 - x^2 + x - 7}{x^3 - x^2 - x - 2} \, dx. \quad [10]$$

14V A coin is tossed n times. Each coin toss is independent and the probability of heads is α and tails is $1 - \alpha$. Let the random variable X denote the number of heads observed across the n trials.

- (a) Write down an expression for $P(k, n, \alpha)$, the probability of observing exactly k heads in total from the n trials. [2]

- (b) Consider

$$E[X(X-1)] = \sum_{k=0}^n k(k-1) \binom{n}{k} \alpha^k (1-\alpha)^{n-k}.$$

- (i) By expanding the binomial coefficient in terms of factorials, or otherwise, show that

$$E[X(X-1)] = n(n-1)\alpha^2 \sum_{k=2}^n \binom{n-2}{k-2} \alpha^{k-2} (1-\alpha)^{n-k}. \quad [3]$$

- (ii) By writing $j = k - 2$, or otherwise, show that $E[X(X-1)] = n(n-1)\alpha^2$. [3]

- (iii) Given that $E[X(X-1)] = E[X^2] - E[X]$, show that $E[X^2] = n\alpha(1-\alpha) + n^2\alpha^2$. (You may state without proof the mean of this distribution.) [3]

- (c) The i^{th} central moment is defined as $\mu_i = E[(X - E[X])^i]$. Show that the second central moment, $\mu_2 = n\alpha(1-\alpha)$. [6]

- (d) For $n = 100$, the variance of X was found to be 24. Find the possible values of α . [3]

15Y

- (a) Find explicitly the solution $y(x)$ of the differential equation

$$x \frac{dy}{dx} - 3y + xy = 0$$

for $x > 0$, subject to the condition

$$y(1) = e. \quad [5]$$

- (b) Consider the differential equation

$$x^2 \frac{dy}{dx} + y^2 - xy = x^2$$

for $x > 0$.

- (i) Find the general solution of the equation, which may be in implicit form. [7]
(ii) Find explicitly the solution of the equation subject to the condition

$$y(1) = 0. \quad [2]$$

- (c) Find explicitly the general solution $y(x)$ of the differential equation

$$x^3 \frac{dy}{dx} - x \sin(2x) + 2x^2 y = 0$$

for $x > 0$. [6]

16T Consider the real-valued function of two real variables x and y defined by

$$f(x, y) = (x^2 - 1)^2 + (y^2 - 1)^2 + x^2 y^2.$$

(a) State the definition of a stationary point of the function $f(x, y)$. [1]

(b) Find the coordinates of all stationary points of $f(x, y)$. [6]

(c) Find algebraic expressions for the elements of the following matrix:

$$\mathbf{H} = \begin{pmatrix} \frac{\partial^2 f(x, y)}{\partial x^2} & \frac{\partial^2 f(x, y)}{\partial x \partial y} \\ \frac{\partial^2 f(x, y)}{\partial y \partial x} & \frac{\partial^2 f(x, y)}{\partial y^2} \end{pmatrix}. \quad [1]$$

(d) By using the elements of the matrix $\mathbf{H}$, classify the stationary points of $f(x, y)$ for $x \geq 0$ and $y \geq 0$ and then, using symmetry arguments, or otherwise, complete the classification of the other stationary points. (Matrix properties may be used, but are not necessary, in order to answer this question.) [5]

(e) Sketch the contours of $f(x, y)$ and add arrows to your sketch showing the directions of the gradient vector $\nabla f(x, y)$ in the neighbourhoods of the stationary points. [7]

17Z

- (a) Cramer's rule for writing down the solutions (if any) of the linear equation $\mathbf{Ax} = \mathbf{y}$ is

$$x_i \det(\mathbf{A}) = \det(\mathbf{A}_i)$$

where $\mathbf{A}_i$ is the matrix $\mathbf{A}$ with the i^{th} column replaced by the vector $\mathbf{y}$ and x_i is the i^{th} element of $\mathbf{x}$.

Consider the system of linear equations representing a pair of lines in the (x_1, x_2) -plane:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = y_1, \\ a_{21}x_1 + a_{22}x_2 = y_2, \end{cases}$$

where $a_{ij}, y_i \in \mathbb{R}$ for $i, j = 1, 2$. Using Cramer's rule, provide the geometric interpretation and the specific restrictions on parameters a_{ij} and y_i for the following cases:

- (i) a unique solution,
- (ii) no solution,
- (iii) multiple solutions.

[5]

- (b) Assuming that the matrix $\mathbf{A}$ is invertible, use your results from (a) to find the inverse of

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

expressing your answer in terms of the matrix elements a_{ij} .

[2]

- (c) Consider an $n \times n$ matrix $\mathbf{B}$ acting on an $n \times 1$ vector $\mathbf{x}$ to produce an $n \times 1$ vector $\mathbf{y}$, with components

$$y_i = B_{ij}x_j.$$

- (i) Show that, for a solution $\mathbf{x}$ to exist, the vector $\mathbf{y}$ must be expressible as a linear combination of the columns of the matrix $\mathbf{B}$.

[Hint: B_{i1} are the elements of the vector that is the first column of $\mathbf{B}$.]

[2]

- (ii) Describe the conditions, in terms of the columns of the general $n \times n$ matrix $\mathbf{B}$, that lead to multiple solutions, a unique solution, or to no solution, and show how your results in (a) illustrate these conditions.

[4]

- (d) Consider the matrix

$$\mathbf{C} = 2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} + 4 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}.$$

Let $\mathbf{y} = \mathbf{Cx}$, where

$$\mathbf{y}^T = (6 \quad \beta \quad \gamma).$$

Solve for $\mathbf{x}$ in terms of the real parameters α , β and γ , identifying and describing all three possible classes of solution.

[7]

18R Let

$$f(x) = x(\pi - |x|) \quad (-\pi < x \leq \pi),$$

and extend f to a 2π -periodic function on $\mathbb{R}$.

(a) Show that f is odd. Hence show that the Fourier series of f has the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx).$$

Compute the Fourier coefficients b_n .

[5]

(b) By evaluating your series at $x = \frac{\pi}{2}$, deduce that

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} = \frac{\pi^m}{n},$$

where m and n are integers that you should determine.

[7]

(c) State Parseval's Theorem for a general 2π -periodic function on $\mathbb{R}$.

[2]

(d) Using your earlier results, or otherwise, show that

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^6} = \frac{\pi^p}{q},$$

where p and q are integers that you should determine.

[6]

19S*

(a) Consider the sum $\sum_{k=1}^n a_k$ subject to the constraint $\sum_{k=1}^n a_k^2 = 1$.

(i) State Schwarz's inequality. [1]

(ii) Compute the largest value that the sum can take subject to the constraint. Show when this maximum can be attained. [3]

(b)

(i) Let $f(x, t)$ be a function such that f and $\partial f / \partial t$ are continuous for $0 \leq x \leq t \leq 1$. Define

$$F(t) = \int_0^t f(x, t) \, dx.$$

By considering $F(t+h) - F(t)$, or otherwise, derive the expression for $\frac{dF}{dt}(t)$. [2]

(ii) Let

$$J(a) = \int_0^a \ln(1+x^2) \, dx \quad (a > 0).$$

Determine whether $J(a)/a$ is an increasing, decreasing or non-monotonic function for $a > 0$. [4]

(c) Let $f(x)$ be a positive, decreasing, and continuous function for $x \geq 1$.

(i) For any integer $n \geq 1$,

$$\int_1^{n+1} f(x) \, dx \leq \sum_{k=1}^n f(k) \leq f(1) + \int_1^n f(x) \, dx.$$

Prove either one of these two inequalities. [2]

(ii) Apply the result of (c)(i) to the function $f(x) = x^{-1/2}$ to show that

$$\sum_{k=1}^n \frac{1}{\sqrt{k}} = 2\sqrt{n} + O(1) \quad \text{as } n \rightarrow \infty. [3]$$

(iii) Calculate the result of the series

$$S(n) = \sum_{k=1}^n (\sqrt{k+1} - \sqrt{k}). [2]$$

(iv) Compute the limit of the k^{th} term of the series in (c)(iii) as $k \rightarrow \infty$. Hence comment briefly on how your result for (c)(iii) is consistent with the approximation obtained in (c)(ii). [*Hint: consider $1/(\sqrt{k+1} + \sqrt{k})$ and its limit as $k \rightarrow \infty$.*] [3]

20Y* Consider a cone with height h and rim of radius r , where h and r are variable. Its curved surface has a fixed area A . The cone is filled with ice cream so that the volume of the ice cream is that of the cone plus an additional solid hemisphere with radius r (see figure below).

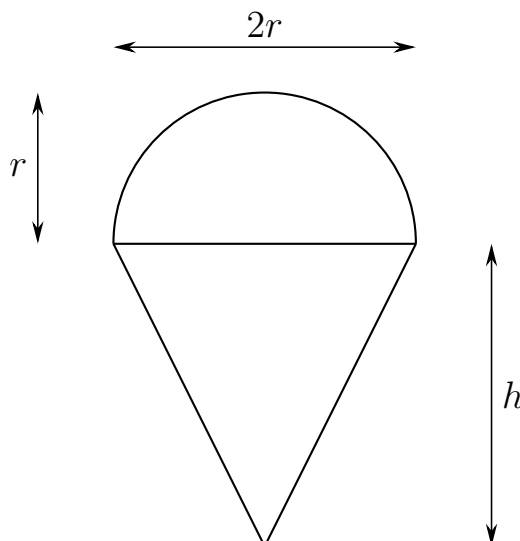
- (a) Using the method of Lagrange multipliers, write the equations whose solution gives the stationary value for the volume of ice cream. [5]

[You may use that the area of the curved surface of a cone is $\pi r\sqrt{r^2 + h^2}$.]

- (b) Hence find the height, radius, and volume of the stationary solution. [10]

- (c) Find the volume in the limits $h \rightarrow 0$ and $h \rightarrow \infty$. Hence, or otherwise, classify the stationary point found in (b). [5]

[You may use that $\frac{\sqrt{11}-1}{(21-6\sqrt{11})^{3/4}} = 2.156\dots$]



END OF PAPER