UNIVERSITY OF CAMBRIDGE

NST2PC/MTH/2

NST1

NATURAL SCIENCES TRIPOS Part IB

Tuesday 17 June 2025 9:00am to 12:00pm

MATHEMATICS (2)

Read these instructions carefully before you begin:

You may submit answers to no more than **six** questions. All questions carry the same number of marks.

The approximate number of marks allocated to a part of a question is indicated in the right-hand margin.

Write on **one** side of the paper only and begin each answer on a separate sheet.

At the end of the examination:

Each question has a number and a letter (for example, **8B**).

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

A **separate** green main cover sheet listing all the questions attempted **must** also be completed.

Every cover sheet must bear your examination number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on top, and then the cover sheet and answer for each question, in the numerical order of the questions.

Calculators and other electronic or communication devices are not permitted in this examination.

STATIONERY REQUIREMENTS
6 gold cover sheets
Green main cover sheet
Script paper
Rough paper
Treasury tag

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1B

Consider the eigenvalue problem

$$u'' + \frac{2}{x}u' = -\lambda\, u \ , \qquad (*)$$

where the function $u(x)$ satisfies the boundary conditions that $u(0)$ is finite and $u(\pi/2) = 0$. Here $u' = du/dx$.

(a) Rewrite $(*)$ in Sturm-Liouville form and determine its weight function. Show explicitly that the Sturm-Liouville operator is self-adjoint for this system. [5]

Show that eigenfunctions $u_a(x)$ and $u_b(x)$ corresponding to distinct eigenvalues satisfy the orthogonality relation

$$\int_0^{\pi/2} x^2\, u_a(x) u_b(x) dx = 0 \ . \qquad [2]$$

(b) By using the change of dependent variable $v(x) = xu(x)$, or otherwise, find the set of eigenfunctions and corresponding eigenvalues to $(*)$. [5]

(c) Consider the differential equation

$$y'' + \frac{2}{x}y' - \beta y = \frac{1}{x}\cos(x) \ ,$$

where $\beta > 0$, and the boundary conditions are that $y(0)$ is finite and $y(\pi/2) = 0$. Using the eigenfunctions of $(*)$, show that the solution to this equation is of the form

$$y(x) = \sum_n a_n u_n(x) \ ,$$

with an appropriate range for the sum and coefficients a_n that you need to determine. [8]

2C

Laplace's equation in spherical coordinates is given by

$$\nabla^2\Psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2} = 0 \qquad (1)$$

(a) Use separation of variables to show that a general solution that is axially symmetric (i.e. symmetric about the axis aligned with $\theta = 0$) is given by

$$\Psi = \sum_{l=0}^{\infty}\left(A_l r^l + B_l r^{-l-1}\right)P_l(\cos\theta), \qquad (2)$$

where P_l is the Legendre polynomial that solves Legendre's equation

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_l(x)}{dx}\right] + l(l+1)P_l(x) = 0 \qquad (3)$$

and A_l and B_l are constants. [6]

(b) The boundary condition on a sphere of radius a is given by

$$\Psi(r=a,\theta,\phi) = 4 + \cos(\theta) - 2\cos(2\theta) - 6\sin^2(\theta).$$

Find the solution for Ψ outside the sphere, assuming that $\Psi \to 0$ as $r \to \infty$. Then find Ψ inside the sphere, assuming that Ψ is finite in the interior. [10]
[*Hint: Recall the Legendre Polynomials* $P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{1}{2}(3x^2-1)$.]

(c) A flat sheet held at $\Psi = 0$ is added outside the sphere on the equator (i.e., at $\theta = \pi/2$). The boundary conditions on the surface of the sphere are unchanged. Find Ψ outside the sphere. [4]

3A

An electrostatic field $\mathbf{E} = (E_x, E_y, E_z)$ in three dimensions satisfies the Maxwell equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \qquad \nabla \times \mathbf{E} = 0.$$

Suppose further that the $\mathbf{E}$ field and the charge density $\rho(x, y)$ are independent of the z-coordinate, and that $|\mathbf{E}| \to 0$ as $y \to \pm\infty$.

(a) Rewrite both Maxwell equations in terms of an electrostatic potential φ. [2]

(b) Show that the most general possible Green's function for the potential φ sourced by a two-dimensional delta function $\rho = \delta(x - x')\delta(y - y')$, subject to the stated boundary conditions on $\mathbf{E}$, is

$$G(x, y; x', y') = A \log((x - x')^2 + (y - y')^2) + C$$

where A is a constant that you will determine, and C is an undetermined constant. [5]

(c) Consider the specific case in which the charge density takes the form

$$\rho = \delta(y)$$

within the interval $x \in (0, L)$, while $\rho = 0$ when x is outside of this interval. Calculate the potential φ. [8]

[*Hint: You may find it useful to recall the integral* $\displaystyle\int \frac{du}{1 + u^2} = \tan^{-1}(u)$.]

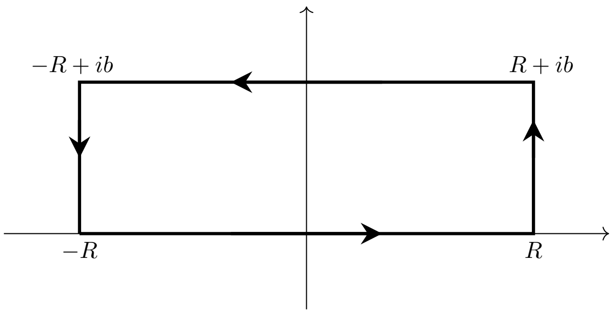
(d) Consider the same charge density ρ as in part (c). Using the solution you found in part (c), or otherwise, verify that when $0 < x < L$, the electric field component E_y jumps discontinuously at $y = 0$, and calculate the size of the discontinuity $|\Delta E_y|$ across $y = 0$. [5]

4B

(a) Derive the integration formula

$$\int_0^\infty e^{-x^2} \cos(2bx) dx = \frac{\sqrt{\pi}}{2} e^{-b^2} \ ,$$

where $b \in \mathbb{R}$, by integrating the function e^{-z^2} around the rectangular path shown in the figure below and taking an appropriate limit. [8]



(b) Consider the function

$$f(z) = \frac{\cos(az)}{z^2 + b^2} \ ,$$

where a and b are real constants. By choosing a suitable closed contour Γ, evaluate

$$I_1 = \int_0^\infty \frac{\cos(ax)}{x^2 + b^2} dx \ . \qquad [6]$$

Based on your findings for I_1, evaluate the following integrals

$$I_2 = \int_0^\infty \frac{x \sin(x)}{x^2 + b^2} dx \ , \qquad [3]$$

and

$$I_3 = \int_0^\infty \frac{\sin(x)}{x(x^2 + b^2)} dx \ . \qquad [3]$$

5C In this problem, you should assume that the Fourier transform of a function $f(t)$ is given by $\tilde{f}(\omega) = \int dt\ f(t)e^{-i\omega t}$.

(a) Consider a particle at position $x(t)$ experiencing a force $F(t)$. The position is determined by the following differential equation:

$$x'' + 5x' + 6x = F(t),$$

where primes indicate derivatives with respect to time.

Using Fourier transform methods, show that the solution can be written in terms of a transfer function $R(\omega)$, which you should specify, as follows:

$$\tilde{x}(\omega) = \tilde{F}(\omega)R(\omega)$$

where $\tilde{x}(\omega)$ and $\tilde{F}(\omega)$ are the Fourier transforms of $x(t)$ and $F(t)$.

Hence find the solution $x(t)$, assuming $x(t \leqslant 0) = 0$, $F(t \geqslant 0) = e^{-t}$, and $F(t < 0) = 0$. [12]

(b) Now assume that the motion of the particle is determined by a different (unknown) ordinary differential equation which is still linear in the coordinate $x(t)$:

$$\sum_{n=1}^{c} a_n \frac{d^n x}{dt^n} = F(t)$$

where each a_n is a constant and c is an integer.

Measurements of $\tilde{x}(\omega)$ are made with a known input force $\tilde{F}(\omega)$, and the transfer function $R(\omega)$ is thus determined. $R(\omega)$ is found to have simple poles at $\omega = 0$, $\omega = 2i$, and $\omega = -i$. Find a set of coefficients a_n that could describe this system, up to an overall multiplicative constant. [8]

6A

[Throughout this question the Einstein summation convention is assumed, and the dimension of space is taken to be three.]

(a) State the transformation law for a tensor of rank-4 under rotations. Write an equation giving the condition for this tensor to be isotropic. [2]

(b) Any rank-2 tensor T_{ij} may be written as:

$$T_{ij} = X_{ij} + A_{ij} + t\delta_{ij}, \qquad (*)$$

where A_{ij} is an antisymmetric tensor, and X_{ij} is a symmetric tensor that is also traceless ($X_{ii} = 0$), and t is a scalar. Verify $(*)$ by solving for X_{ij}, A_{ij}, and t in terms of T_{ij}. [4]

(c) With X_{ij} and A_{ij} defined as in part (b), show that the following contractions all vanish:

(i) $X_{ij}A_{ij}$; [2]

(ii) $A_{ij}A_{jk}A_{ki}$; [3]

(iii) any product of an odd number of A_{ij} tensors, where all indices are contracted to form a scalar. [3]

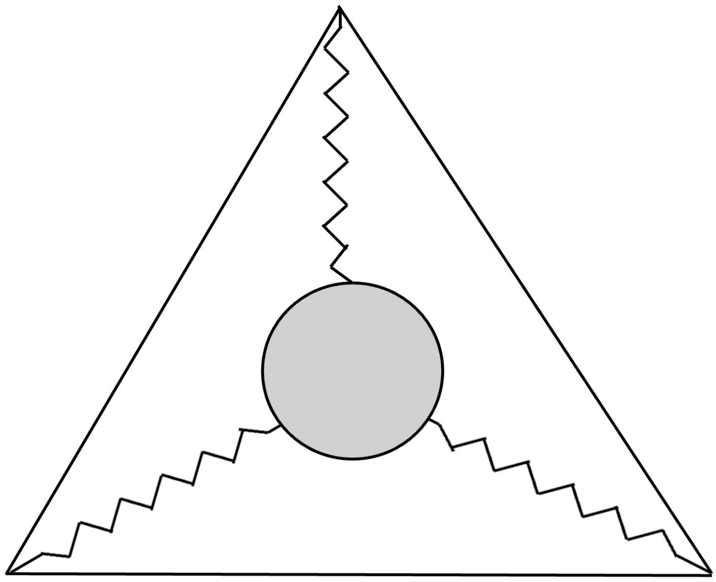
(d) Let M_{ij} be a constant rank-2 tensor. Consider the following three-dimensional integral:

$$I = \int_B dx_1\, dx_2\, dx_3\, M_{ij}x_ix_j$$

over the ball region B consisting of points inside the unit sphere ($r^2 = x_1^2 + x_2^2 + x_3^2 \leqslant 1$). Using rotational symmetry, or otherwise, evaluate the integral I. [6]

7C

A uniform disk of mass m and radius a lies on a horizontal, frictionless surface. It is attached symmetrically to three ideal, massless springs with spring constant k, which are in turn connected to three corners of an equilateral triangle, as illustrated in the figure below. The triangle is fixed and cannot move. In equilibrium, the length of all springs l is greater than their length at rest l_0.



(a) Write down the Lagrangian for this system, assuming that any displacements or rotations of the disk from the equilibrium position are small.
[*Hint: You may use the fact that the rotational kinetic energy of the disk is given by* $\frac{1}{2}I(\dot{\theta})^2$, *where* $I = \int dA\ \rho r^2$, ρ *is density per area, and* θ *is the angle of rotation.*] [8]

(b) Find all the normal modes of the system and their frequencies. Sketch the normal modes. [7]

(c) Instead of attaching the disk with springs to the corners of an equilateral triangle, now consider attaching the disk with springs to the corners of a regular pentagon (i.e., one with equal sides and angles). The five springs are again attached symmetrically to the disk, and the pentagon is again fixed in place. How many normal modes does this system have and how many of these would you expect to always be degenerate (i.e. have the same frequency of oscillation)? Briefly and qualitatively justify your answer. [5]

8A

(a) Consider the following multiplication tables, for a binary operation $*$ or $\times$ acting on a set with 5 elements:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	a	1	c	d	b
b	b	d	1	a	c
c	c	b	d	1	a
d	d	c	a	b	1

$\times$	1	a	b	c	d
1	1	a	b	c	d
a	a	c	d	b	1
b	b	d	a	1	c
c	c	b	1	d	a
d	d	1	c	a	b

For each table, show that it either forms a group, or does not form a group (as the case may be). [6]

(b) Let us define a *gloop* as a nonempty set that is closed under an associative binary operation, satisfying these axioms (in place of the usual identity and inverse axioms):

1. Each element a has an element a_L^{-1}, such that $ba_L^{-1}a = b$ for all elements b, and
2. Each element a has an element a_R^{-1}, such that $aa_R^{-1}b = b$ for all elements b.

Show that in fact a gloop is a group. [Do not assume anything not stated explicitly above.] [8]

(c) Consider the symmetry group of a regular hexagon (6 sides), including both rotations and reflections. What is the order of this group? Partition the elements into conjugacy classes, and indicate the number of elements in each class. [6]

9A

(a) State briefly what is meant by the terms *group homomorphism*, *group isomorphism*, and *kernel*. [3]

(b) Show that any group of order 101 is abelian. [You may assume without proof Lagrange's Theorem.] [5]

(c) Consider the cyclic group of order 6, $G = \mathbb{Z}_6$. Identify all of the homomorphisms from the group G to the 2 element cyclic group $\mathbb{Z}_2$, and to the 3 element cyclic group $\mathbb{Z}_3$. [6]

(d) Besides $\mathbb{Z}_6$, the only other group of order 6 is $G = \Sigma_3$, the permutation group acting on a 3 element set. For this group, again identify all of the homomorphisms from the group G to $\mathbb{Z}_2$, and to $\mathbb{Z}_3$. [6]

10B Consider a finite group G with order $|G|$, and let d_α be an irreducible representation of G. We define

$$\nu_\alpha \equiv \frac{1}{|G|} \sum_{g \in G} \chi_\alpha(g^2) \ ,$$

where χ_α is the character of the representation.

(a) Write a two-dimensional irreducible representation for the cyclic group Σ_3, and evaluate ν_α. [5]

(b) Write a two-dimensional irreducible representation for the Vierergruppe C_4, and evaluate ν_α. [5]

(c) Given an irreducible representation d_α, we define

$$A \equiv \sum_{g \in G} D(g^2) \ .$$

For a two-dimensional irreducible representation of the cyclic group Σ_3, evaluate A. [You may use the same representation as before.] Show that your answer takes the form

$$A = \frac{|G|\nu_\alpha}{|d_\alpha|} I \ ,$$

where I is the identity element in the representation and $|d_\alpha|$ is the dimension of the representation. [5]

For a two-dimensional irreducible representation of the Vierergruppe C_4, evaluate A. [You may use the same representation as before.] Show that your answer again takes the form

$$A = \frac{|G|\nu_\alpha}{|d_\alpha|} I \ . \qquad [5]$$

END OF PAPER