UNIVERSITY OF CAMBRIDGE

NST2PC/MTH/1

NST1

NATURAL SCIENCES TRIPOS Part IB

Monday 2 June 2025 9:00am to 12:00pm

MATHEMATICS (1)

Read these instructions carefully before you begin:

You may submit answers to no more than **six** questions. All questions carry the same number of marks.

The approximate number of marks allocated to a part of a question is indicated in the right-hand margin.

Write on **one** side of the paper only and begin each answer on a separate sheet.

At the end of the examination:

Each question has a number and a letter (for example, **8B**).

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

A **separate** green main cover sheet listing all the questions attempted **must** also be completed.

Every cover sheet must bear your examination number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on top, and then the cover sheet and answer for each question, in the numerical order of the questions.

Calculators and other electronic or communication devices are not permitted in this examination.

STATIONERY REQUIREMENTS
6 gold cover sheets
Green main cover sheet
Script paper
Rough paper
Treasury tag

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1C

(a) Show that

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b},$$

where $\mathbf{a}$ and $\mathbf{b}$ are three-dimensional vector fields. [5]

(b) State the divergence theorem and use it to prove the following identity:

$$\int_V [\mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})] \, dV = \int_S (\mathbf{F} \times \mathbf{G}) \cdot \hat{\mathbf{n}} \, dS,$$

where $\mathbf{F}$ and $\mathbf{G}$ are three-dimensional vector fields, V is a volume enclosed by the closed surface S, $\hat{\mathbf{n}}$ is the outward unit vector normal to S. [8]

(c) A vector field is given by $\mathbf{A}(\mathbf{x}) = (x^3 + 2y + z^2, y^3, 0)$. Let $\tilde{S}$ be the surface defined by

$$x^2 + y^2 = 1 + z^2,$$

where $-1 < z < 1$. [7]

Compute $\int_{\tilde{S}} \mathbf{A} \cdot \hat{\mathbf{n}} \, dS$, where $\hat{\mathbf{n}}$ is the outward unit vector normal to $\tilde{S}$.

[*Hint: You may wish to use the divergence theorem.*]

2C

The diffusion of Uranium-235 neutrons is governed by a modified diffusion equation for the neutron number density n, which is a function of time t and three-dimensional position $\mathbf{x} = (x, y, z)$:

$$\frac{\partial n}{\partial t} = \mu \nabla^2 n + \gamma n$$

where γ and μ are positive constants. Note that the number of neutrons is not conserved.

Consider a block of uranium in the shape of a cube with side length L. One corner of the cube lies at the origin; the cube edges are oriented along the positive coordinate axes so that all points in the cube have positive or zero x, y, and z coordinates. You may assume that everywhere on the boundary of the cube, the density of neutrons is $n = 0$.

(a) Using the method of separation of variables, show that the number density of neutrons can be described by a mode expansion

$$n = \sum_{p,q,r=1}^{\infty} a_{p,q,r} T_{p,q,r}(t) X_p(x) Y_q(y) Z_r(z)$$

where $a_{p,q,r}$ is a constant and you should determine the functions $T_{p,q,r}(t)$, $X_p(x)$, $Y_q(y)$, and $Z_r(z)$. [9]

(b) Assuming fixed values of μ and γ and a generic initial neutron distribution, find the critical side-length of the cube L_{crit} such that the number density of neutrons will increase exponentially with time if (and only if) $L > L_{\text{crit}}$. [4]

(c) Now assume that the block of uranium is allowed to depart from a cubic geometry, giving a cuboid with different side-lengths. In particular, assume that the side-length of the block in the x direction, L_x, is now allowed to vary by a factor λ from that of a cube, so that $L_x = \lambda L$. The other two side-lengths are correspondingly reduced to keep the volume unchanged, with $L_y = L_z = L/\sqrt{\lambda}$. Write down the solution for the neutron density for this cuboid. [No justification is needed.] If L is twice the critical side-length for the cube, show that there are values of λ at which the neutron density no longer grows exponentially. [7]

3B Consider the following second order differential equation

$$L\,u(x) = f(x)\ , \quad \text{with} \quad L\,u(x) = \frac{d^2u}{dx^2} + \frac{n+1}{x}\frac{du}{dx}\ , \qquad (\star)$$

for a function $u(x)$ defined on $1/2 \leqslant x \leqslant 1$, n an integer, and $f(x)$ a smooth function. The boundary conditions on $u(x)$ are $u(1/2) = 0$ and $u(1) = 0$.

(a) Construct the homogenous solutions to L for $n \neq 0$, and separately for $n = 0$. [4]

(b) Define and construct the Green's function associated with these boundary conditions. Write the general solution to $L\,u(x) = f(x)$ in terms of the Green's function. [Analyse the case with $n \neq 0$ and $n = 0$ separately.] [7]

(c) For $n \neq 0$, use the Green's function and evaluate the appropriate integral, to obtain a solution to $(\star)$ when

$$f(x) = \frac{1}{x}\ , \qquad \frac{1}{2} \leqslant x \leqslant 1\ .$$

Identify which value of n you need to treat separately. Check explicitly that your final answer satisfies the appropriate boundary conditions. [9]

4C The Fourier transform of a function $f(x)$ is given by $\tilde{f}(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx} f(x)$.

(a) Assume that $h(x) = \int_{-\infty}^{\infty} dx' f(x - x') g(x')$. Prove the convolution theorem for the Fourier transforms $\tilde{h}, \tilde{f}, \tilde{g}$ of the functions h, f, g:

$$\tilde{h}(k) = \tilde{f}(k)\tilde{g}(k).$$ [5]

(b) Consider the function:

$$g(x) = \begin{cases} 1, & \text{if } |x| < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find its Fourier transform $\tilde{g}(k)$. [4]

(c) Derive the Fourier transform of the function $u(x)$, which is defined by

$$u(x) = \begin{cases} 2 - |x|, & \text{if } |x| < 2, \\ 0, & \text{otherwise.} \end{cases}$$

[*Hint: It may be helpful to construct this function from convolutions of g defined as in part (b).*] [4]

(d) Consider the function $r(x) = \sum_{n=1}^{4} u(x + 4n - 10)$, where u is defined as in part (c). Compute the Fourier transform of this function, $\tilde{r}(k)$. Then evaluate

$$\int_{-\infty}^{\infty} dk' e^{i(20k')} \tilde{g}(k') \tilde{r}(k - k'),$$

with $\tilde{g}(k)$ defined as in part (b). [7]

5B

(a) Find the matrix that transforms the standard basis of $\mathbb{C}^3$ to the vectors

$$\mathbf{u}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} \\ \frac{1+i}{\sqrt{6}} \end{pmatrix} , \quad \mathbf{u}_2 = \begin{pmatrix} -\frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{6}} \\ -\frac{1-i}{\sqrt{6}} \end{pmatrix} , \quad \mathbf{u}_3 = \begin{pmatrix} 0 \\ -\frac{2}{\sqrt{6}} \\ \frac{1+i}{\sqrt{6}} \end{pmatrix} . \qquad [2]$$

Show that this matrix is unitary. [2]

(b) Let A and B be $n \times n$ Hermitian and invertible matrices. [You may assume, without proof, standard properties of Hermitian matrices.]

(i) Show that the product AB is Hermitian if and only if $AB = BA$. [1]

(ii) Find a number c (real or complex) such that $AB + cBA$ is Hermitian. [1]

(iii) Show that the product ABA is Hermitian. [1]

(iv) Show that $U = (A + iB)(A - iB)^{-1}$ is unitary if $AB = BA$. [3]

(v) Consider a unitary matrix C. Show that $(C^{-1}AC)^p$ has real eigenvalues, where p is a positive integer. [3]

(vi) Consider

$$H \equiv A^2 + \alpha A + \beta \mathbb{I} , \quad \alpha, \beta \in \mathbb{R} , \quad \alpha^2 < 4\beta .$$

Here $\mathbb{I}$ is the identity matrix. Show that H cannot have a zero eigenvalue. What would go wrong if $\alpha^2 \geqslant 4\beta$? [7]

6B

(a) Prove that if a matrix C satisfies $C^\dagger C = 0$, then $C = 0$. Argue why $C^2 = 0$ does not imply that C is zero by finding a nonzero 2×2 matrix whose square is zero. [3]

(b) For which complex numbers c do there exist $n \times n$ matrices M, N such that $MN - NM = c\mathbb{I}$ where $\mathbb{I}$ is the identity matrix? Explain your answer. [2]

(c) A and B are $n \times n$ invertible matrices. Show that if $AB + BA = 0$, then n must be even and $\mathrm{Tr}A = \mathrm{Tr}B = 0$. [*Hint: You might want to consider the eigenvalues of A and B.*] [5]

(d) The exponential of a $n \times n$ matrix P is defined as

$$\exp(P) \equiv \mathbb{I} + \sum_{k=1}^{\infty} \frac{P^k}{k!} \ ,$$

with $\mathbb{I}$ the $n \times n$ identity.

(i) Consider the following matrix functions

$$f_1(t) = \exp(tP)\exp(tQ) \ , \qquad f_2(t) = \exp(t(P+Q)) \ ,$$

where P and Q are $n \times n$ matrices and $PQ = QP$. Show that f_1 and f_2 satisfy the differential equation

$$\frac{df_i}{dt} = f_i(t)(P+Q) \ , \qquad i = 1, 2 \ .$$

Also show that $f_1(t = 0) = f_2(t = 0)$. Use this to argue that

$$\exp(P)\exp(Q) = \exp(P+Q) \ ,$$

when $PQ = QP$. [5]

(ii) Show that

$$\exp(tR)S\exp(-tR) = S + t[R,S] + \frac{t^2}{2}[R,[R,S]] + O(t^3)$$

where R and S are $n \times n$ matrices, and t is a real parameter. In this expression, brackets are defined as

$$[A,B] \equiv AB - BA \ ,$$

where A and B are $n \times n$ matrices. [5]

7C

An analytic function of a complex number $f(z)$ can be written as $f(z) = u(x, y) + iv(x, y)$ where u and v are real functions and $z = x + iy$.

(a) State and prove the Cauchy-Riemann equations. Hence show that contours of constant u and constant v are perpendicular at every point. [5]

(b) Find the analytic function $f(z)$ that has the real part $u(x, y) = e^x(x \cos y - y \sin y)$. You may assume that $f(0) = 0$. [6]

(c) Find and classify the zeros and singularities (including any at infinity) of the functions

$$\frac{e^z}{(z^2+4)z} \quad \text{and} \quad \frac{z^2}{\sin^2(z)}. \qquad [3]$$

(d) Find the Laurent series expansion of $\dfrac{1}{(z-i)(z+3)}$ about $z = 1$ and determine its radius of convergence. [6]

8A

Consider the following second-order differential equation for a function $x(y)$:

$$x + x' + (y^2 - 1)x'' = 0.$$

where the prime represents the d/dy derivative.

(a) Identify all singular points at finite values of y. Which of them are regular? [4]

(b) Find the roots σ_1, σ_2 of the indicial equation at $y = 1$, and write down the corresponding power series solutions. [6]

(c) Determine the recurrence relation for the coefficients of both series solutions, and calculate the first subleading coefficient in each series. [10]

9A

(a) Consider the following action for a particle moving in one dimension:

$$S = \int_{-\infty}^{+\infty} dt \left[-m\sqrt{1-\dot{x}^2} - V(x) \right].$$

Here the dot represents the d/dt time derivative, $m > 0$ is a constant, and the speed $|\dot{x}| < 1$ for all physical trajectories. Using the Euler-Lagrange equation, determine the equation of motion for the particle. Using the fact that the Lagrangian has no dependence on t except via $x(t)$, write down a first integral (i.e. conserved quantity) for this action, and show that it takes the form:

$$\frac{m}{\sqrt{1-\dot{x}^2}} + V(x) = \text{const.}$$ [8]

(b) With the same setup as in part (a), suppose now that the potential depends only on the sign of x, so that:

$$V(x) = \begin{cases} V_1, & x < 0, \\ V_2, & x > 0, \end{cases}$$

with V_1, V_2 constant, and $V_2 > V_1$. [This should be regarded as the limiting case of a potential which very steeply increases from V_1 to V_2 in the vicinity of $x = 0$.] Suppose further that the particle starts at $x = -1$ with an initial velocity of $\dot{x} = v_i$ at the initial time t_i, where $0 < v_i < 1$. Determine the threshold value of v_i required for the particle to pass to the other side of $x = 0$. Calculate the final velocity v_f of the particle, as a function of v_i, both above and below this threshold. [12]

10B

Consider a one-dimensional particle modelled by following differential equation

$$-\frac{d^2\psi}{dx^2} - V_0 e^{-x^2}\psi = E\psi \ , \qquad V_0 > 0 \ , \qquad (\star)$$

where E is the energy of the particle. The particle is subject to the boundary condition $\psi \to 0$ as $|x| \to \infty$. The lowest energy of this system is denoted by E_0.

(a) Show that finding the values of E is equivalent to finding the extrema of functional

$$\Lambda[y] = \frac{F[y]}{G[y]} \ ,$$

where

$$F[y] = \int_{-\infty}^{\infty} \Big((y')^2 - V_0 e^{-x^2} y^2 \Big)\, dx \ , \qquad G[y] = \int_{-\infty}^{\infty} y^2 dx \ .$$

With this, explain how to estimate E_0 by using $\Lambda[y]$. [5]

(b) For the trial function

$$y_{\rm T}(x) = e^{-\frac{1}{2}ax^2} \ ,$$

where a is a positive real parameter, show that

$$E(a) \equiv \Lambda[y_{\rm T}] = \frac{1}{2}a - V_0\sqrt{\frac{a}{1+a}} \ .$$

State briefly why $y_{\rm T}(x)$ is good choice of trial function. [4]

(c) Find an equation that characterises the extrema of $E(a)$. Find the zeros of $E(a)$. Sketch the behaviour of $E(a)$ for $a \geqslant 0$. From here show that the lowest energy eigenvalue E_0 should be negative. [7]

(d) For $0 < V_0 \ll 1$, show that

$$E_0 \leqslant -\frac{1}{2}V_0^2 + \epsilon$$

where $\epsilon = O(V_0^4)$. [4]

END OF PAPER