UNIVERSITY OF CAMBRIDGE

CST0/MATH/2
CEBT0/5

NST0

NATURAL SCIENCES TRIPOS Part IA

Wednesday 18 June 2025 9:00 am to 12:00 pm

MATHEMATICS (2)

Read these instructions carefully before you begin:

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

You may submit answers to **all** of section A, and to no more than **five** questions from section B.

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on one side of the paper only and begin each answer on a separate sheet. (For this purpose, your section A attempts should be considered as one single answer.)

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Complete a gold cover sheet for your section A answer, and place it at the front of your answer to section A.

Complete a gold cover sheet for **each** section B answer, and place it at the front of your answer to that question.

A **separate** green main cover sheet listing all the questions attempted **must** also be completed. (Your section A answer should be recorded just as A: there is no need to list each individual short question.)

Every cover sheet must bear your blind grade number and desk number, and should NOT include your name or CRSid.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
6 gold cover sheets.
Green main cover sheet.
Script paper.
Treasury tag.

SPECIAL REQUIREMENTS
No calculators may be used.
No electronic devices may be used.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1 (a) Calculate the vector area for the triangle with vertices A, B, C given by position vectors $\mathbf{a} = (1,0,0)^\intercal$, $\mathbf{b} = (0,1,0)^\intercal$ and $\mathbf{c} = (0,0,1)^\intercal$, travelling round the perimeter in the direction $A \to C \to B \to A$. [1]

(b) For the vector field $\mathbf{F} = (1,1,-1)^\intercal$ calculate the flux through the triangle assuming the same direction for the path on the perimeter as in (a). [1]

2 Do the unit vectors $\frac{1}{3}(2,2,1)^\intercal$, $(0,0,1)^\intercal$ and $\frac{1}{\sqrt{3}}(1,1,1)^\intercal$ provide a basis in three dimensions? Justify your answer. [2]

3 Consider the linear equations for x and y defined by $\begin{pmatrix} -1 & 1 \\ 1 & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 2y \end{pmatrix}$.

(a) Write down an equation for b such that non-zero solutions exist for (x, y). [1]

(b) Solve the equation for b. [1]

4 Find the Fourier series on $-\pi \leqslant x < \pi$ for the function $f(x) = \cos(3x)\cos(5x)$. [2]

5 Assuming summation over repeated indices, rewrite the following suffix expressions in terms of the vectors $\mathbf{a}, \mathbf{b}$ and the matrices $\mathbf{C}, \mathbf{D}$, where a_i are the components of $\mathbf{a}$ and C_{ij} are the elements of $\mathbf{C}$ etc.

(a) $\delta_{ij} a_i b_j$, [1]

(b) $C_{ij} D_{ij}$. [1]

6 Find the complementary function for the ordinary differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - 6y = 1\,. \qquad [2]$$

7 A sphere of unit radius is centred at the origin. The density is $\rho(x, y, z) = \rho_0 \sqrt{x^2 + y^2}$. Calculate the mass of the sphere. [2]

8 For the vector field $\mathbf{F} = (xy^2, 0, 0)^\intercal$ calculate:

(a) $\operatorname{div}\mathbf{F}$ (i.e. $\boldsymbol{\nabla}\cdot\mathbf{F}$), [1]

(b) $\operatorname{curl}\mathbf{F}$ (i.e. $\boldsymbol{\nabla}\times\mathbf{F}$). [1]

9 Let λ be an eigenvalue of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

(a) Write down the characteristic equation for λ. [1]

(b) Solve the equation for λ. [1]

10 A fair coin is tossed three times. Calculate the probabilities for:

(a) three heads, [1]

(b) precisely two tails. [1]

SECTION B

11T

Consider the equation

$$az\bar{z} + b\bar{z} + \bar{b}z + c = 0 \; , \qquad (\dagger)$$

where $b = b_1 + ib_2$ and $z = x + iy$ and $\bar{z}$ is the complex conjugate of z and where a, b_1, b_2, c are real constants and x, y are real variables.

(a) Let $a = 0$ and $b \neq 0$.

(i) Without writing z in terms of x and y, describe the locus of z in the Argand plane. [1]

(ii) Write Eq. ($\dagger$) as a real equation in terms of x and y. [1]

(b) Let $a = 1$ and $b \neq 0$.

(i) Without writing z in terms of x and y, describe the locus of z in the Argand plane. [3]

(ii) Write Eq. ($\dagger$) as a real equation in terms of x and y. [3]

(iii) What is the constraint on the parameters b and c which must be satisfied for real solutions of Eq. ($\dagger$) to exist? [1]

(iv) Under what circumstances, if any, is Eq. ($\dagger$) satisfied by a single point? [1]

(c) Show that the following transformations of Eq. ($\dagger$) map it to an equation of the form

$$Aw\bar{w} + B\bar{w} + \bar{B}w + C = 0 \, .$$

For each transformation, find A, B and C in terms of a, b, c and α.

(i) Rotation: $z = \alpha w$ with $|\alpha| = 1$ and $\alpha \in \mathbb{C}$. [2]

(ii) Dilation: $z = \alpha w$ with $\alpha \in \mathbb{R}$ and $\alpha > 0$. [3]

(iii) Translation: $z = \alpha + w$ with $\alpha \in \mathbb{C}$. [3]

(iv) Inversion: $z = \dfrac{1}{w}$ for $w \neq 0$. Identify the conditions on the coeffcients in Eq. ($\dagger$) so that $A = 0$. [2]

12V

A bag contains g green balls and r red balls (and no others). A game is defined by the following procedure: a ball is drawn from the bag and placed on the table; this is repeated (without replacement) until the number of tabled green balls is equal to the number of tabled red balls, or all the balls have been tabled. A game is successful if it ends because the same number of green and red balls have been tabled, and is unsuccessful otherwise.

(a) What is the probability that the first three balls drawn are all green? You may assume $g \geqslant 3$. [2]

(b) Assuming sufficiently many green and red balls in the bag, what is the probability that the game ends successfully after exactly x balls have been drawn where

(i) $x = 1$? [1]

(ii) $x = 2$? [2]

(iii) $x = 4$? [2]

You may leave your answer as a product of fractions in each case.

(c) Assuming sufficiently many green and red balls in the bag, how many different orders of drawing green and red balls result in a successful game drawing a total of y balls where

(i) $y = 4$? [1]

(ii) $y = 6$? [2]

(d) To play a different game with the $g + r$ balls, the numbers 1 to g are written on the green balls and 1 to r are written on the red balls. It is known that $r \geqslant g$. A ball is drawn at random, found to have x written on it, and put back in the bag. What is the probability that drawing a second ball will find a number less than or equal to x? *[Hint: you might find it useful to consider separate cases.]* [5]

(e) Suppose $r = g = 10$ and that the red balls are renumbered from 6 to 15 (the green balls are still labelled 1 to 10). A ball with value x is drawn from the bag and not replaced.

(i) What is the probability that the next ball is green given that $x = 4$? [1]

(ii) What is the probability that the next ball is green given that $x = 8$ and that the first ball was green? [2]

(iii) What is the probability that the next ball is green given that $x = 8$? [2]

13Y

(a) Consider the pair of simultaneous ordinary differential equations for real-valued functions $x(t)$ and $y(t)$:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \beta y + \sin t \,, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = x + \cos t \,, \end{cases}$$

where the parameter $\beta \in \mathbb{R}$.

(i) Rewrite these equations to obtain a second-order differential equation for $y(t)$. [2]

(ii) Determine the general solution for $y(t)$ and its dependence on β. Find all possible functional forms of the general solution $y(t)$ of the second-order differential equation derived in (i) and specify the values of β for which they are valid. [6]

(b) Consider the second order differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - 6\frac{\mathrm{d}y}{\mathrm{d}t} + 9y = 3e^{3t}(t+1)\,.$$

Find its solution $y(t)$ subject to boundary conditions $y(0) = 1$ and $y(1) = 0$. [12]

14T

(a) Consider the equation

$$\frac{\partial^2 u(x,y)}{\partial x^2} - \frac{1}{2}\frac{\partial u(x,y)}{\partial y} - y\frac{\partial^2 u(x,y)}{\partial y^2} = 0 \qquad (\dagger)$$

defined for $x, y \in \mathbb{R}$ with $y > 0$.

Let $\xi(x,y) = x + 2\sqrt{y}$ and $\eta(x,y) = x - 2\sqrt{y}$. Using this change of variables find expressions for the following first and second partial derivatives in terms of the new coordinates ξ, η and the partial derivatives of $u(\xi, \eta)$ with respect to ξ and/or η:

(i) $\dfrac{\partial u}{\partial x}$, [4]

(ii) $\dfrac{\partial u}{\partial y}$, [4]

(iii) $\dfrac{\partial^2 u}{\partial x^2}$, [4]

(iv) $\dfrac{\partial^2 u}{\partial y^2}$. [4]

(b) Using the results of (a) write Eq. (†) in terms of ξ and η. [4]

15V

(a) Evaluate the double integral

$$I = \iint_D \cos x \cos y \, e^{\sin y} \, \mathrm{d}x \, \mathrm{d}y$$

in Cartesian coordinates, where D is defined by the inequalities $0 \leqslant x \leqslant \pi$ and $0 \leqslant y \leqslant x$. [4]

(b) Evaluate the triple integral

$$J = \iiint_S x|y| \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

in Cartesian coordinates for each of the following domains, where S is defined by the inequalities:

(i) $0 \leqslant x \leqslant 1, \quad 0 \leqslant y \leqslant x, \quad 0 \leqslant z \leqslant xy$; [8]

(ii) $0 \leqslant x \leqslant 1, \quad -x/2 \leqslant y \leqslant x, \quad 0 \leqslant z \leqslant e^y$. [8]

16S

(a) Consider the vector field $\mathbf{F}(x, y) = (y^2 - 2x, 2xy - y)^\intercal$ and the curve C parameterised by $\mathbf{r}(t) = (t^2, t^3)^\intercal$, with $t \in [0, 1]$.

(i) Compute the line integral of $\mathbf{F}$ along the curve C. [5]

(ii) If $\mathbf{F}$ is conservative, find a potential $\phi(x, y)$ such that $\mathbf{F} = \boldsymbol{\nabla}\phi$. If not, explain why a potential does not exist. [5]

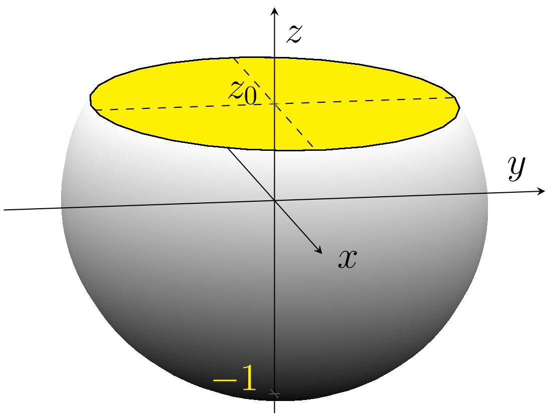
(b) Let $\mathbf{F}(\mathbf{r}) = (P(x, y, z), Q(x, y, z), R(x, y, z))^\intercal$ be a vector field in $\mathbb{R}^3$, and let $\mathbf{r}(t)$ be a parameterised curve given by $\mathbf{r}(t) = (x(t), y(t), z(t))^\intercal$, where $t \in [a, b]$. Derive the expression for

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(\boldsymbol{\nabla} \times \mathbf{F}) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right] .$$ [10]

You may assume that all derivatives appearing in your analysis exist.

17W

A sphere of unit radius is centred at the origin and truncated. The surface of the resulting closed shape consists of two parts: the curved spherical surface with $-1 \leqslant z < z_0$ and the flat horizontal top at $z = z_0$ where $-1 < z_0 < 1$. A particular case of the unit sphere truncated at a positive value of z_0 is shown in the figure below.



The outward flux, $\Phi(z_0) = \iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$, of the vector field $\mathbf{F}$ through the closed surface S of the truncated sphere depends on the coordinate z_0 of truncation.

(a) For $\mathbf{F} = \mathbf{r} \equiv (x, y, z)^\intercal$,

(i) find $\Phi(z_0)$, [8]

(ii) sketch the graphs of $\Phi(z_0)$ and $\dfrac{\mathrm{d}\Phi(z_0)}{\mathrm{d}z_0}$ versus z_0. [4]

(b) For $\mathbf{F} = \dfrac{\mathrm{d}f(z)}{\mathrm{d}z}\mathbf{r}$, where $f(z)$ is a differentiable function of a single variable z coordinate only,

(i) find $\Phi(z_0)$ in terms of $f'(z_0)$, $f(z_0)$, $f(-1)$ and some other function of z_0, [6]

(ii) find a function $f(z)$ which gives the same answer for the flux $\Phi(z_0)$ as that obtained in (a)(i). [2]

18Z

(a) Consider a real-valued $n \times n$ matrix $\mathbf{M}$, where n is a positive integer. Show that it can be decomposed into $\mathbf{M} = \mathbf{S} + \mathbf{A}$, where $\mathbf{S}$ is symmetric and $\mathbf{A}$ is antisymmetric, giving explicit expressions for both $\mathbf{S}$ and $\mathbf{A}$ in terms of $\mathbf{M}$. [2]

(b) For a symmetric $n \times n$ matrix $\mathbf{S}$, where n is a positive integer, it is given that all of its eigenvalues λ_i are non-zero real numbers, and the corresponding eigenvectors $\mathbf{u}_i$ are orthogonal.

(i) An arbitrary vector $\mathbf{x}$ can be represented with respect to a basis of orthogonal vectors $\{\mathbf{u}_i\}$ as

$$\mathbf{x} = \sum_{i=1}^{n} \alpha_i \mathbf{u}_i \,.$$

Give expressions for the coefficients $\{\alpha_i\}$ in terms of $\mathbf{x}$ and $\mathbf{u}_i$ and hence find an expression for the coefficients β_i in the formula

$$\mathbf{S}\mathbf{x} = \sum_{i=1}^{n} \beta_i \mathbf{u}_i \,.$$ [3]

(ii) Define a new matrix $\mathbf{C}$ by the Taylor series expansion

$$\mathbf{C} = \sum_{p=0}^{\infty} \frac{1}{p!} \mathbf{S}^p \,,$$

where $\mathbf{S}^0 \equiv \mathbf{I}$ is the identity matrix. By considering the action of the matrix $\mathbf{C}$ on $\mathbf{u}_i$, find the eigenvectors and corresponding eigenvalues of $\mathbf{C}$. [3]

(iii) Write down the eigenvalues and eigenvectors of $\mathbf{S}^{-1}$ in terms of the eigenvalues and eigenvectors of $\mathbf{S}$. [2]

(iv) For the matrix $\mathbf{S} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ write the vector $\mathbf{x} = (1, 0)^\intercal$ in terms of the eigenvectors of $\mathbf{S}$, and hence find

$$\mathbf{y} = \mathbf{S}^{-1}\mathbf{x} \,,$$

leaving your answer in terms of the eigenvectors of $\mathbf{S}$. [3]

(c) Consider a non-zero 2×2 anti-symmetric matrix $\mathbf{A}$ with $(\mathbf{A})_{12} = a$ being a real number.

(i) Show that the action of $\mathbf{A}$ on any non-zero two-dimensional vector is a rotation of this vector by $\pi/2$ combined with another geometric action. What is this geometric action? What is the result of applying $\mathbf{A}$ twice? [3]

(ii) Define a new matrix $\mathbf{D}$ by the Taylor series expansion

$$\mathbf{D} = \sum_{p=0}^{\infty} \frac{1}{p!} \mathbf{A}^p \,.$$

Show that

$$\mathbf{D} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} ,$$

giving θ in terms of a. [4]

19R*

(a) By using ε-δ notation:

(i) define what is meant by saying that a real-valued function $f(x)$ defined for all real x is continuous at the point x_0, [1]

(ii) define what is meant by saying that $f(x)$ is differentiable at x_0. [1]

(b) Let

$$f_a(x) = \begin{cases} |x|^a \sin(\pi \sin(1/x)) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0\,, \end{cases}$$

where a is a real parameter. Consider the sequences $x_n = 1/(2n\pi + \pi/6)$ and $x'_n = 1/(2n\pi)$, for positive integer n. Evaluate $f_0(x_n)$ and $f_0(x'_n)$. [2]

(c) By using the definitions of continuous and differentiable functions, give and justify your answers to the following questions.

(i) For what values of a, if any, is $f_a(x)$ continuous at $x = 0$? [5]

(ii) For what values of a, if any, is $f_a(x)$ differentiable at $x = 0$? [6]

[Hint: to prove discontinuity and/or non-differentiability for some values of a it may be helpful to consider the values of f_a on the sequences given above.]

(d) By stating and using standard tests for convergence or divergence, determine if the series $\sum_{n=1}^{\infty} f_{-1}(n)$ is convergent or divergent. [5]

20Z*

The wave equation may be written as

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}\,,$$

valid for $-\infty < x < \infty$ and $-\infty < t < \infty$, where c is a positive constant.

(a) By considering the variables $\eta = x + ct$ and $\xi = x - ct$, show that the wave equation can be recast as

$$\frac{\partial^2\psi}{\partial\eta\partial\xi} = 0\,,$$

and hence deduce that the general solution will be of the form

$$\psi(x,t) = f(x+ct) + g(x-ct)\,,$$

for any twice-differentiable functions f and g. [8]

(b) We now impose initial conditions

$$\psi(x,0) = u(x)\,,$$
$$\frac{\partial\psi}{\partial t}(x,0) = v(x)\,.$$

Give expressions for $f(x)$ and $g(x)$ in terms of the function $u(x)$ and an integral of the function v. Hence give an expression for $\psi(x,t)$. [7]

(c) Find $\psi(x,t)$ explicitly for the case

$$u(x) = 0\,,$$
$$v(x) = 2cxe^{-\frac{x^2}{2}}\,,$$

and sketch $\psi(x,t)$ as a function of x at t_1 and t_2 with $0 < t_1 < t_2$. [5]

END OF PAPER