UNIVERSITY OF CAMBRIDGE

CST0/MATH/1
CEBT0/4

NST0

NATURAL SCIENCES TRIPOS Part IA

Monday 16 June 2025 9:00 am to 12:00 pm

MATHEMATICS (1)

Read these instructions carefully before you begin:

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

You may submit answers to **all** of section A, and to no more than **five** questions from section B.

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on one side of the paper only and begin each answer on a separate sheet. (For this purpose, your section A attempts should be considered as one single answer.)

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Complete a gold cover sheet for your section A answer, and place it at the front of your answer to section A.

Complete a gold cover sheet for **each** section B answer, and place it at the front of your answer to that question.

A **separate** green main cover sheet listing all the questions attempted **must** also be completed. (Your section A answer should be recorded just as A: there is no need to list each individual short question.)

Every cover sheet must bear your blind grade number and desk number, and should NOT include your name or CRSid.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS
6 gold cover sheets.
Green main cover sheet.
Script paper.
Treasury tag.

SPECIAL REQUIREMENTS
No calculators may be used.
No electronic devices may be used.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1 Solve for real x,

$$4^x - 2^x - 2 = 0\,. \qquad [2]$$

2 Given that the perimeter and area of a regular n-sided polygon inscribed in a unit circle are equal to $mn \sin \frac{\pi}{n}$ and $\frac{n}{k} \sin \frac{2\pi}{n}$, respectively, where m and k are integers, find m and k. [2]

3 The sum of two non-zero position vectors $\mathbf{u} = (a, b, a+b)^\intercal$ and $\mathbf{v} = (4a, -b, a-b)^\intercal$, with a and b being real parameters, is the zero vector. Find:

(a) the value of a, [1]

(b) the angle between vectors $\mathbf{u}$ and $\mathbf{v}$. [1]

4 Let

$$f(x) = \left(\frac{x}{a}\right)^2 + \left(\frac{a}{x}\right)^2\,,$$

where a is a positive parameter.

(a) For what value of a does the function $f(x)$ have a stationary point at $x = 1$? [1]

(b) Sketch the graph of $f(x)$ for $a = 2$. [1]

5 Evaluate the following integrals:

(a)

$$\int (\cos^2 x - 1)\,\mathrm{d}x\,, \qquad [1]$$

(b)

$$\int_{\pi/3}^{\pi/6} \frac{1}{\cos^2 x}\,\mathrm{d}x\,. \qquad [1]$$

6 Evaluate:

(a)
$$-\frac{1}{2}+\frac{3}{2}-\frac{5}{2}+\frac{7}{2}-\ldots+\frac{95}{2}-\frac{97}{2}+\frac{99}{2}\,,$$ [1]

(b)
$$\sum_{n=3}^{\infty}\left(\frac{1}{3}\right)^{n}\,.$$ [1]

7 The real-valued function $y(x)$ obeys the following differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=y-1\,.$$

Find $y(1)$ if:

(a) $y(0)=0\,,$ [1]

(b) $y(0)=1\,.$ [1]

8 Find the values of a, b and c in the following expression:

$$\frac{x^2+1}{x^2-3x+2}=a+\frac{b}{x-1}+\frac{c}{x-2}\,.$$ [2]

9 Two circles are defined by the following equations:

$$\begin{aligned}(x-1)^2+(y-1)^2 &= 1\,,\\ (x+1)^2+(y-1)^2 &= 1\,.\end{aligned}$$

Find:

(a) the coordinates of a point belonging to both circles, [1]

(b) the distance between the centres of the circles. [1]

10 An unbiased 6-sided die is thrown twice and the scores are recorded separately.

(a) What is the probability that the sum of the two scores is less than 4? [1]

(b) What is the probability that the magnitude of the difference between the two scores equals 1? [1]

SECTION B

11S

(a) Let x, y, z be Cartesian coordinates in three-dimensional space. Consider the equation

$$x + 2y - 3z = 0\,.$$

(i) What object is described by this equation? [1]

(ii) Find two linearly independent vectors in this object. [1]

(iii) Can one find three linearly independent vectors in this object? Justify your answer. [1]

(b) Consider the equation

$$|\mathbf{r} - \hat{\mathbf{n}}\,(\mathbf{r}\cdot\hat{\mathbf{n}})| = 2\,, \qquad (\dagger)$$

where $\mathbf{r} = (x, y, z)^{\mathsf{T}}$ and $\hat{\mathbf{n}} = \frac{1}{\sqrt{2}}(1, 0, 1)^{\mathsf{T}}$.

(i) Rewrite this equation in Cartesian coordinates. [2]

(ii) Does Eq. (†) describe a line, a plane, a sphere, a cone, a cylinder or something else? State which. [1]

(iii) Consider the plane given by

$$x + z = 0\,. \qquad (\ddagger)$$

What is the relation between the normal to the plane and $\hat{\mathbf{n}}$? What is the shape of the intersection of this plane and the object described by Eq. (†)? [4]

(iv) Given the point with position vector $\mathbf{p} = (2, \sqrt{2}, 0)^{\mathsf{T}}$, compute its shortest distance to the plane given by Eq. (‡). [3]

(c) Consider the two lines given by:

$$L_1 : \ \mathbf{r}_1 = \mathbf{a}_1 + \lambda_1 \mathbf{v}_1\,,$$
$$L_2 : \ \mathbf{r}_2 = \mathbf{a}_2 + \lambda_2 \mathbf{v}_2\,,$$

where $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^3$ are two position vectors, $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^3$ are two non-zero vectors and $\lambda_1, \lambda_2 \in \mathbb{R}$ are real parameters.

(i) Compute the shortest distance between the lines L_1 and L_2 in terms of the scalar products and vector products of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{v}_1$ and $\mathbf{v}_2$, when $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. [3]

(ii) What is the shortest distance between the lines L_1 and L_2 when $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent? [2]

(iii) If $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, which conditions do $\mathbf{a}_1, \mathbf{a}_2, \mathbf{v}_1$ and $\mathbf{v}_2$ have to satisfy so that the distance between the lines L_1 and L_2 is zero, and what is the geometrical interpretation of these conditions? [2]

12R

(a) Give the first two non-zero terms of the Taylor series expansions of the following functions:

(i) $\tanh(x)$ around $x = 0$, using any method, [6]

(ii) $(2^x - 2^{-x})\sin(x)$ around $x = 0$, using any method. [6]

(b) By calculating directly the derivatives of the function $\ln(e^{ax} - b)$ evaluated at $x = 1$, give the first two non-zero terms of its Taylor series expansion around $x = 1$. Here the function is defined for the variable $x > 0$ and the parameters a and b obey the inequalities $a > 0$ and $0 < b < 1$. For this function, you should distinguish between possible cases arising for different values of a and b in the given ranges. [8]

13X

Calculate the following integrals. Show your working as well as listing your solutions. You may quote and use standard integrals.

(a)
$$\int_0^1 x^2\, e^{\frac{x}{2}}\, \mathrm{d}x$$ [4]

(b)
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2\, \sin x \, \mathrm{d}x$$ [2]

(c)
$$\int \frac{\mathrm{d}x}{x(x^2+1)}$$ [4]

(d)
$$\int \frac{1+x}{\sqrt{1+x^2}}\, \mathrm{d}x$$ [4]

(e)
$$\int \frac{1}{5 + 3\cos x}\, \mathrm{d}x$$ [6]

14V

The continuous random variable X has the following probability distribution

$$f(x) = \begin{cases} 0 & x < -\pi/6 \ , \\ ax + b\cos x & -\pi/6 \leqslant x \leqslant \pi/6 \ , \\ 0 & x > \pi/6 \ , \end{cases}$$

where a and b are real parameters.

(a) By using the general properties of probability distributions, find:

(i) the value of b, [2]

(ii) the range of possible values of a. [4]

(b) Find a if the expectation value of X is $E[X] = \frac{2}{3}\left(\frac{\pi}{6}\right)^3$. [4]

(c) If $a = 0$, find the variance of the random variable X. [6]

(d) The continuous random variable Y has probability density

$$g(y) = \begin{cases} 0 & y < -\pi/6 \ , \\ P(X \leqslant y)/M & -\pi/6 \leqslant y \leqslant \pi/6 \ , \\ 0 & y > \pi/6 \ , \end{cases}$$

where P denotes the probability. Find M. [4]

15Y

(a) Find explicitly the general solution $y(x)$ of the following differential equations:

(i)
$$y \sin x - \cos x \frac{\mathrm{d}y}{\mathrm{d}x} = 0\,,$$ [4]

(ii)
$$x^2 + y^2 - xy\frac{\mathrm{d}y}{\mathrm{d}x} = 0\,.$$ [6]

(b) Determine whether the differential form

$$x\left(y^4+1\right)\mathrm{d}x + y\left(2x^2y^2+1\right)\mathrm{d}y$$

is exact. Hence, or otherwise, solve explicitly the following equation for $u(x)$:

$$x\left(u^2+1\right) + \frac{1}{2}\left(2x^2u+1\right)\frac{\mathrm{d}u}{\mathrm{d}x} = 0\,,$$

subject to the boundary condition $u(1) = 0$. [10]

16T

Consider

$$f(x,y) = (x-1)\ (y+1)\ \exp\left(-\frac{(x-1)^2+(y+1)^2}{2}\right)\,.$$

(a) Find the coordinates of the stationary points of $f(x,y)$. [5]

(b) Find the values of $f(x,y)$ at the stationary points. [5]

(c) Sketch the contours of $f(x,y)$ and determine the types of the stationary points. [6]

(d) Add arrows to your sketch showing the directions of the gradient vectors $\boldsymbol{\nabla} f(x,y)$ near the stationary points. [4]

17Z

(a) Using suffix notation, show that for any three $D \times D$ matrices $\mathbf{P}, \mathbf{Q}$ and $\mathbf{R}$ the trace of $\mathbf{PQR}$ is equal to the trace of $\mathbf{QRP}$, where D is a positive integer. [2]

(b) For the vector $\mathbf{n} = (n_1, n_2)^\intercal$, state in terms of n_1 and n_2 the result of

(i) $\mathbf{n}\mathbf{n}^\intercal$, [1]

(ii) $\mathbf{n}^\intercal\mathbf{n}$. [1]

(c) Consider the 3×3 matrix

$$\mathbf{M} = (\mathbf{I} - \hat{\mathbf{n}}\hat{\mathbf{n}}^\intercal) \ ,$$

where $\hat{\mathbf{n}}$ is a real-valued column vector of unit length, and $\mathbf{I}$ is the 3×3 identity matrix. What is the geometric action of $\mathbf{M}$ on an arbitrary real 3-dimensional column vector $\mathbf{x}$? [3]

(d) The matrix $\mathbf{M}$ given in part (c) can be written in suffix notation as

$$M_{ij} = \delta_{ij} - \hat{n}_i\hat{n}_j .$$

Find the trace of $\mathbf{M}$. [1]

(e) What is the result of the action of the matrix $\mathbf{P}$, where $P_{ij} = \delta_{ij} - p_i p_j$ and $\mathbf{p} = (1, 1, 0)^\intercal$, on an arbitrary vector $\mathbf{x} = (x_1, x_2, x_3)^\intercal$? Give a geometric interpretation of this result. [3]

(f) Now consider the matrix $\mathbf{Q}$ where $Q_{ij} = \delta_{ij} - q_i q_j$ with $\mathbf{q} = (1, -1, 0)^\intercal$. Find the vector $\mathbf{y}$, which is the result of the action of the matrix $\mathbf{PQ}$ on an arbitrary vector $\mathbf{x} = (x_1, x_2, x_3)^\intercal$, i.e.

$$\mathbf{y} = \mathbf{PQx}.$$

Give a geometrical interpretation of this result. [5]

(g) Find a matrix $\mathbf{R}$ such that the combined effect of $\mathbf{N} = \mathbf{PQR}$ on an arbitrary vector $\mathbf{x}$ is $\mathbf{Nx} = -\mathbf{x}$. Give your answer in the form

$$R_{ij} = \delta_{ij} - r_i r_j ,$$

specifying the (constant) vector $\mathbf{r}$. [4]

18R

(a) Let n be an integer.

(i) Using integration by parts, evaluate

$$\int_0^{\pi} e^x \cos(nx)\, \mathrm{d}x$$

as an expression involving $\int_0^{\pi} e^x \sin(nx)\, \mathrm{d}x$. [1]

(ii) Similarly, using integration by parts, evaluate

$$\int_0^{\pi} e^x \sin(nx)\, \mathrm{d}x$$

as a different expression involving $\int_0^{\pi} e^x \cos(nx)\, \mathrm{d}x$. [1]

(iii) Hence, or otherwise, evaluate both integrals. [2]

(b) Explain what is meant by the Fourier series of a periodic function $f(x)$ with period $2L$, giving expressions for the coefficients of the series. [4]

(c) The function $f(x)$ is periodic, with period 2π, and is defined for $-\pi \leqslant x < \pi$ by

$$f(x) = e^{|x|}\,.$$

Find the Fourier series for $f(x)$. [7]

(d) By using the Fourier series for $f(x)$ at $x = \pi/2$, show that

$$\frac{\pi}{4\sinh(\pi/2)} - \frac{1}{2} = \sum_{k=1}^{\infty} (-1)^k \frac{1}{mk^2+1}\,,$$

where you need to determine the value of the integer m. [5]

19S*

Investigate the limit as the positive integer $n \to \infty$ of

$$a_n = \frac{\sqrt{n}}{n!}\left(\frac{n}{e}\right)^n$$

by following the steps:

(a) Show

$$\lim_{n\to\infty} a_n = \exp\left(-1 + \lim_{n\to\infty}\sum_{k=1}^{n} u_k\right), \qquad [3]$$

where $u_n = \ln(a_{n+1}/a_n)$. You may assume that

$$\lim_{n\to\infty} \ln(a_n) = \ln\left(\lim_{n\to\infty} a_n\right).$$

(b) Prove that the necessary condition for convergence of the series for partial sums of u_n is obeyed. [3]

(c) By expanding u_n as a Taylor series in $1/n$ for $n > 1$, demonstrate that

$$u_n = \sum_{k=2}^{\infty}(-1)^k \frac{1}{n^k}\left(\frac{1}{k+1} - \frac{1}{2k}\right). \qquad [3]$$

(d) Prove that the u_n are positive. [5]

(e) Find an upper bound for u_n in the form of a geometric series and thus demonstrate that $u_n < An^m$, for some integer m and some positive parameter A whose values you should state. [4]

(f) Determine whether the sequence (a_n) converges. [2]

20Y*

The temperature T at a point on the surface of an ellipsoid is given by

$$T(x, y, z) = x^4 + (yz)^2 \,,$$

where the surface of the ellipsoid is defined by

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2 + \left(\frac{z}{b}\right)^2 = 1 \,,$$

and a and b are positive parameters.

(a) Using the method of Lagrange multipliers, write the equations whose solutions give the coordinates of the extremum temperature points on the ellipsoid. Hence, or otherwise, find the temperature at the coldest points on the ellipsoid and find their coordinates. [8]

(b) Find the temperatures at and coordinates of any other points, besides those with the lowest temperature, where the temperature is stationary on the ellipsoid. [10]

(c) In the case where the ellipsoid is the unit sphere (i.e. $a = b = 1$), find the temperature at the hottest points on the ellipsoid and find their coordinates. [2]

END OF PAPER