NST1

NATURAL SCIENCES TRIPOS Part IB

Tuesday 11 June 2024 9:00am to 12:00pm

MATHEMATICS (2)

Read these instructions carefully before you begin:

You may submit answers to no more than **six** questions. All questions carry the same number of marks.

The approximate number of marks allocated to a part of a question is indicated in the right-hand margin.

Write on **one** side of the paper only and begin each answer on a separate sheet.

At the end of the examination:

Each question has a number and a letter (for example, **8B**).

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

A **separate** green main cover sheet listing all the questions attempted **must** also be completed.

Every cover sheet must bear your examination number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on top, and then the cover sheet and answer for each question, in the numerical order of the questions.

Calculators and other electronic or communication devices are not permitted in this examination.

STATIONERY REQUIREMENTS
6 gold cover sheets
Green main cover sheet
Script paper
Rough paper
Treasury tag

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1C The concentration $c(x,t)$ of a substance diffusing in one dimension with spatially varying (positive) diffusivity $D(x)$, in the presence of a (bounded) external potential $V(x)$, obeys

$$\frac{\partial c}{\partial t} = -\frac{\partial J}{\partial x}\,, \text{ where } \quad J = -D(x)\frac{\partial c}{\partial x} - \beta D(x) c\,\frac{dV}{dx}\,, \qquad (\dagger)$$

with β a positive constant. The substance is confined to the region $-L < x < L$ and the boundary conditions are that $J = 0$ at $x = \pm L$.

(a) Show that $c(x,t) = u(x)e^{-\lambda t}$ is a solution, where $u(x)$ and λ obey

$$\frac{d}{dx}\left[D(x)\frac{du}{dx} + \beta D(x) u\,\frac{dV}{dx}\right] = -\lambda u\,. \qquad (\ddagger)$$ [3]

(b) Confirm that the change of variable $u(x) = e^{-\beta V(x)} y(x)$ reduces ($\ddagger$) to the Sturm-Liouville eigenvalue problem

$$\mathcal{L}y \equiv -(py')' + qy \quad ; \quad \mathcal{L}y = \lambda w y\,, \qquad (\S)$$

and give the resulting $p(x), q(x)$ and $w(x)$. [4]

(c) Show explicitly that the resulting $\mathcal{L}$ is self-adjoint, with respect to the inner product $\langle f|g\rangle = \int_L^L f^*(x)g(x)dx$, for any choice of boundary conditions that ensure $g'f^* = f^{*\prime}g$ at each boundary. Here f and g are any two complex solutions y of (§) and * denotes complex conjugate.

Show further that the boundary conditions $J = 0$ in (†) do ensure this. [5]

(d) Show that real eigenfunctions $f_n(x), f_m(x)$ of $\mathcal{L}$ with different eigenvalues obey

$$\int_{-L}^{L} f_n(x) w(x) f_m(x)\, dx = 0\,.$$ [3]

(e) Using the following variational principle (which you are not asked to prove),

$$\lambda_0 = \min_{y(x)} \left[\frac{\int y(x)\mathcal{L}y(x)\,dx}{\int w(x) y(x)^2\,dx}\right]\,,$$

show that the lowest eigenvalue $\lambda = \lambda_0$ is zero for this problem. Find the corresponding eigenfunction f_0 and hence a solution to $\partial c/\partial t = 0$ in (†). [5]

2C Laplace's equation in plane polar coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2}=0\,.$$

Consider $\Phi(r,\theta)$ within a domain comprising the sector of a disc defined by $0<\theta<2\pi\alpha$ with $\alpha<1$, and $0<r<R$. You are told that Φ vanishes at $\theta=0$ and at $\theta=2\pi\alpha$, and that it is nonsingular at $r=0$.

(a) Using separation of variables, find the general solution in the form

$$\Phi=\sum_{n>0}A_n r^{\nu(n)}h_n(\theta)\,,$$

where both $\nu(n)$ and $h_n(\theta)$ should be given explicitly. [6]

(b) Now suppose Φ obeys the additional boundary condition

$$\Phi(R,\theta)=\varphi(\theta)\,,$$

with $\varphi(\theta)$ a known function (itself obeying $\varphi(0)=\varphi(2\pi\alpha)=0$). Give an expression for the coefficients A_n in terms of $\varphi(\theta)$. Is the resulting solution unique? [5]

(c) Rather than being told the precise form of $\varphi(\theta)$, you are now given instead the following information concerning reflection symmetry of the boundary condition, and the behaviour of Φ on the symmetry axis $\theta=\pi\alpha$:

$$\begin{aligned}\varphi(\theta) &= \varphi(2\pi\alpha-\theta)\,,\\ \Phi(r,\pi\alpha) &= \sinh\left(\left(\frac{r}{R}\right)^{1/2\alpha}\right)\,, \qquad r\leqslant R\,.\end{aligned}$$

Find $\Phi(r,\theta)$ everywhere inside the sector. You may use without proof the result

$$\sinh x=\sum_{n=1,3,5\ldots}\frac{x^n}{n!}\,.$$ [9]

3C (a) Define the *fundamental solution* to Poisson's equation $\nabla^2\Phi = \delta(\mathbf{r} - \mathbf{r}')$ in two dimensions. Show that it is given by

$$G_0(\mathbf{r};\mathbf{r}') = A \ln|\mathbf{r} - \mathbf{r}'| + B\,,$$

where A is a constant that you should derive, and B is an arbitrary constant. [3]

(b) State the definition of the Green function $G(\mathbf{r};\mathbf{r}')$ for Poisson's equation with Dirichlet boundary conditions in a two dimensional region S with boundary C. [2]

(c) State Green's identity and hence show that if

$$\begin{aligned} \nabla^2\Phi(\mathbf{r}) &= 0\ ,\ \mathbf{r} \text{ inside } S\,, \\ \Phi(\mathbf{r}) &= f(\mathbf{r})\ ,\ \mathbf{r} \text{ on } C\,, \end{aligned}$$

then

$$\Phi(\mathbf{r}') = \int_C f(\mathbf{r}) \frac{\partial G(\mathbf{r};\mathbf{r}')}{\partial n}\, dC\,. \qquad (*)$$

Here the notation means

$$\frac{\partial G(\mathbf{r};\mathbf{r}')}{\partial n} \equiv \mathbf{n} \cdot \frac{\partial G(\mathbf{r};\mathbf{r}')}{\partial \mathbf{r}}$$

with $\mathbf{n}$ the outward normal to the boundary C of S. [4]

(d) Consider points $\mathbf{r}'$ and $\mathbf{p}(\mathbf{r}')$ related by $\mathbf{p}(\mathbf{r}') = \mathbf{r}'/|\mathbf{r}'|^2$. Show that $|\mathbf{r} - \mathbf{r}'| = |\mathbf{r}'||\mathbf{r} - \mathbf{p}|$ for all points $\mathbf{r}$ that lie on the unit circle, $|\mathbf{r}| = 1$. [2]

(e) Using the method of images or otherwise, show that for the domain S exterior to the unit circle, with Dirichlet boundary conditions on that circle and $G(\mathbf{r}, \mathbf{r}')$ bounded as $|\mathbf{r}| \to \infty$, the Green's function obeys

$$G(\mathbf{r};\mathbf{r}') = A \ln\left(\frac{|\mathbf{r} - \mathbf{r}'|}{|\mathbf{r}'||\mathbf{r} - \mathbf{p}(\mathbf{r}')|}\right)$$

where A has the same value as in (a). [3]

(f) Now suppose that $f(\mathbf{r})$ in $(*)$ is nonzero only in an infinitesimal neighbourhood around $\mathbf{r} = (x, y) = (-1, 0)$ and has $\int f(\mathbf{r})\, dC = 1$. Find $\Phi(\mathbf{r}')$ at a general point $\mathbf{r}' = (x', 0)$ that lies on the x-axis exterior to the unit circle (*i.e.*, $|x'| > 1$). [6]

4A

(a) Prove Cauchy's integral theorem, which states that

$$\oint_C f(z)\, dz = 0$$

for any function f which obeys the Cauchy-Riemann equations throughout the interior of the closed contour C. [4]

(b) What does it mean for a function $f(z)$ to have a *simple pole* at $z = z_0$? State the residue formula. [4]

(c) Consider the function $f(z) = e^{-ikz}\ \mathrm{sech}(z)$, where k is a real constant. Show it has only simple poles, and find their residues. [5]

(d) By relating it to a suitable contour integral, show that Fourier transform

$$\int_{-\infty}^{\infty} e^{-ikx}\ \mathrm{sech}(x)\, dx = \pi\, \mathrm{sech}(k\pi/2)\,.$$ [7]

5A

(a) Define the *convolution* of two functions $f(t)$ and $g(t)$. Show that the Fourier transform of this convolution is the product of the Fourier transforms of $f(t)$ and of $g(t)$. [4]

While updating the college's wiring, an electrician attaches a device to the new wires. In response to a current $I(t)$, the device reads an output $\phi(t)$ determined by

$$\ddot{\phi}(t) + 2p\dot{\phi}(t) + (p^2 + q^2)\phi(t) = I(t)\,, \qquad (\dagger)$$

where (p, q) are real positive constants. Initially $\phi(0) = 0$ and $\dot{\phi}(0) = 0$.

(b) Show that the Fourier transform of the output is given by $\tilde{\phi}(\omega) = \tilde{R}(\omega)\tilde{I}(\omega)$, where $\tilde{I}(\omega)$ is the Fourier transform of the current and

$$\tilde{R}(\omega) = \frac{1}{2iq}\left[\frac{1}{i\omega + p - iq} - \frac{1}{i\omega + p + iq}\right].$$ [4]

(c) The electrician tests the wires by sending current pulse of the form

$$I(t) = \begin{cases} I_0 & \text{if } \; 0 < t < 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

Show that

$$\phi(t) = \frac{I_0}{p^2 + q^2}\left[1 - e^{-pt}\cos(qt) - \frac{p}{q}e^{-pt}\sin(qt)\right]$$

when $t < 1$, and find the form of $\phi(t)$ when $t > 1$. [8]

(d) Use a Fourier transform to solve the homogeneous equation

$$\ddot{\phi}(t) + 2ir\dot{\phi}(t) + (q^2 - r^2)\phi(t) = 0\,,$$

where r is real. [4]

6A

(a) What does it mean for the quantities $T_{i_1 i_2 \cdots i_n}$ to transform as a *tensor* of rank n? What does it mean for a tensor to be *isotropic*? [4]

(b) Let B be the unit ball $x^2 + y^2 + z^2 \leqslant 1$. Show that

$$S_{i_1 i_2 \cdots i_n} = \int_B x_{i_1} x_{i_2} \cdots x_{i_n} \, dV$$

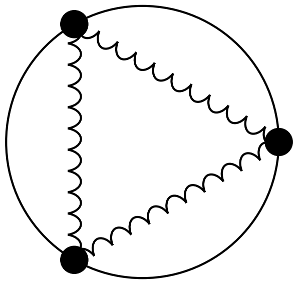
is an isotropic tensor for any positive integer n. Hence evaluate $S_{i_1 i_2}$ and $S_{i_1 i_2 i_3}$. [*You may state without proof the general form of isotropic tensors of rank 2 and 3.*] [7]

A homogeneous but anisotropic crystal has conductivity tensor σ_{ij} that is invariant under rotations around an axis in the direction of the unit vector $\mathbf{n}$. The electric current density $\mathbf{J}$ that flows through the crystal in response to an applied electric field $\mathbf{E}$ has components $J_i = \sigma_{ij} E_j$.

(c) Construct the most general form of σ_{ij} for this crystal. *[Hint: σ_{ij} should be built from tensors invariant under rotations around* $\mathbf{n}$*.]* Give all the circumstances in which $\mathbf{J}$ is parallel to $\mathbf{E}$, stating any additional symmetries of σ_{ij} that are needed in each case. [8]

(d) For a certain crystal of this type, it is observed that no current flows when the electric field points along $\mathbf{n}$. What is the most general form of σ_{ij} for this crystal? [1]

7A A simple toy consists of three identical beads, each of mass m, which slide without friction around a rigid hoop of unit radius. As in the figure, each pair of beads is joined by a spring, all with the same spring constant k and natural length $\sqrt{3}$.



(a) Write down the Lagrangian for the system, assuming the hoop is horizontal and held fixed. Show that the system is in equilibrium when the particles are equally spaced around the hoop. [6]

(b) Find the normal modes and their corresponding frequencies. [10]

(c) Briefly explain the pattern of frequencies you have found, with reference to the symmetry of the toy. [4]

8B Consider a finite group G with elements $g \in G$.

(a) Define the order of the group. Define the order of a group element. [2]

(b) State Lagrange's theorem. Show that if the order of a group is prime then that group has no proper subgroups. [3]

(c) Prove that the order of any group element is a factor of the group's order. [3]

(d) Show that if the order of a group is prime then that group is cyclic. [3]

(e) Show that there is only one group of order three. [3]

(f) Consider two finite groups G_1 and G_2. The set $G_1 \times G_2$ is given by the set of all ordered pairs:

$$G_1 \times G_2 = \{(g_1, g_2) \,|\, g_1 \in G_1 \text{ and } g_2 \in G_2\} \ .$$

We define the product operation

$$(g_1, g_2)(g_1', g_2') \equiv (g_1 g_1', g_2 g_2') \qquad \forall\ (g_1, g_2), (g_1', g_2') \in G_1 \times G_2 \ .$$

Show that with this product operation $G_1 \times G_2$ is a group. What is the order of the group? Show that $G_1 \times I_2$ is a normal subgroup of $G_1 \times G_2$, where I_2 is the identity element of G_2. [6]

9B

(a) Let G be a finite group, with H and K subgroups of G.

Show that the intersection $H \cap K$ is a group. [4]

(b) For two groups G_1, G_2, the *direct product* group $G_1 \times G_2$ is defined by pairs of elements (g_1, g_2), with $g_1 \in G_1$ and $g_2 \in G_2$. The rule for the product of two elements in $G_1 \times G_2$ is given by

$$(g_1, g_2)(g_1', g_2') = (g_1 g_1', g_2 g_2') \ .$$

Show that $Z_2 \times Z_4$ is not isomorphic to Z_8. [5]

Show that $Z_2 \times Z_3$ is isomorphic to Z_6. [5]

What condition do the integers n and m have to satisfy in order for $Z_n \times Z_m$ to be isomorphic to $Z_{n \times m}$? Briefly justify your answer. [2]

[*Here Z_n is the cyclic group of order n.*]

(c) The centre Z of the group G is defined as the set of elements of z in G such that $z = gzg^{-1}$ for all $g \in G$.

Prove that Z is an Abelian subgroup of G. [4]

10B

(a) Consider two finite groups G and H, which are related by the map $\Phi : G \longrightarrow H$.

Give the definition for the map Φ to be a homomorphism. [1]

What are the conditions on G, H and Φ for the map to be an isomorphism? [1]

(b) Let $K = \{I, -I, a, -a\}$ be a multiplicative group of order 4, where $a^2 = -I$.

Write the multiplication table for K. [2]

Consider the map $\Phi : K \longrightarrow GL(2, \mathbb{C})$ such that

$$\Phi(a) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} .$$

Determine $\Phi(I)$, $\Phi(-I)$ and $\Phi(-a)$ such that Φ is a representation of K. [6]

(c) Consider the multiplicative quaternion group $Q = \{\pm I, \pm \boldsymbol{i}, \pm \boldsymbol{j}, \pm \boldsymbol{k}\}$, where

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -I .$$

Show that

$$I \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad \boldsymbol{i} \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \quad \boldsymbol{j} \to \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} , \quad \boldsymbol{k} \to \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

gives rise to a representation of Q in $GL(2, \mathbb{C})$. Is this a faithful or unfaithful representation? Briefly justify your answer. [6]

Evaluate the characters of this representation. [1]

Determine all subgroups of Q, and indicate the order of each subgroup. [3]

END OF PAPER