UNIVERSITY OF CAMBRIDGE

NST2PC/MTH/1

NST1

NATURAL SCIENCES TRIPOS Part IB

Monday 27 May 2024 9:00am to 12:00pm

MATHEMATICS (1)

Read these instructions carefully before you begin:

You may submit answers to no more than **six** questions. All questions carry the same number of marks.

The approximate number of marks allocated to a part of a question is indicated in the right-hand margin.

Write on **one** side of the paper only and begin each answer on a separate sheet.

At the end of the examination:

Each question has a number and a letter (for example, **8B**).

Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.

A **separate** green main cover sheet listing all the questions attempted **must** also be completed.

Every cover sheet must bear your examination number and desk number.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on top, and then the cover sheet and answer for each question, in the numerical order of the questions.

Calculators and other electronic or communication devices are not permitted in this examination.

STATIONERY REQUIREMENTS
6 gold cover sheets
Green main cover sheet
Script paper
Rough paper
Treasury tag

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1C (a) State the *divergence theorem* for a vector field $\mathbf{u}(x, y, z)$ in $\mathbb{R}^3$, defining all the symbols you use. [3]

(b) By setting $\mathbf{u} = \mathbf{a}\psi$, where $\mathbf{a}$ is a constant vector and ψ is a scalar field, show

$$\int_V \boldsymbol{\nabla}\psi \, dV = \int_S \psi \, d\mathbf{S} \,. \qquad [3]$$

(c) By setting $\mathbf{u} = \mathbf{a} \times \mathbf{A}$, where $\mathbf{a}$ is a constant vector and $\mathbf{A}$ is a vector field, show

$$\int_V \boldsymbol{\nabla} \times \mathbf{A} \, dV = \int_S \hat{\mathbf{n}} \times \mathbf{A} \, dS \,,$$

where $\hat{\mathbf{n}}$ is the unit outward normal to the surface S enclosing the volume V. [3]

(d) Consider the vector field $\mathbf{P}$ defined via $\mathbf{P} = \mathbf{r}\,(\boldsymbol{\nabla} \cdot \mathbf{w})$ where $\mathbf{w}$ is another vector field and $\mathbf{r} = (x, y, z)$ is the position vector. Confirm that $P_i = \nabla_j(r_i w_j) - w_i$ and hence establish that for any scalar field $\psi(\mathbf{r})$,

$$\int_V r_i \nabla^2 \psi \, dV = \int_S r_i \nabla_j \psi \, dS_j - \int_S \psi \, dS_i \,. \qquad [3]$$

(e) Similarly show that for any tensor field $\Sigma_{kj}(\mathbf{r})$,

$$\int_V r_i \nabla_j \nabla_k \Sigma_{kj} \, dV = \int_S r_i \nabla_k \Sigma_{kj} \, dS_j - \int_S \Sigma_{ki} \, dS_k \,. \qquad [4]$$

(f) By finding a suitable Σ_{kj}, or otherwise, show that

$$\int_V \mathbf{r} \; \boldsymbol{\nabla} \cdot \left((\nabla^2 \psi) \boldsymbol{\nabla} \psi \right) \, dV$$

can be written as a surface integral for any scalar field $\psi(\mathbf{r})$. [5]

2C In a simple 1D model of a bacterial colony within a region $x \in (0, L)$, the cell density $\rho(x,t)$ obeys

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial x^2} + \gamma\rho \, ,$$

where D is a constant diffusivity and γ is a constant cell division rate. The boundary conditions at the region edges are $\rho(0,t) = \rho(L,t) = 0$ (such that cells leaving the region by diffusion never return).

(a) Using separation of variables, find a general solution in the form

$$\rho(x,t) = \sum_{n=1}^{\infty} a_n \psi_n(x) e^{-\lambda_n t} \, ,$$

where appropriately normalized eigenfunctions ψ_n and eigenvalues λ_n should be found explicitly. [5]

(b) By finding the coefficients a_n, construct the solution with initial condition $\rho(x,0) = \rho_0 \delta(x - x_0)$ where $x_0 \in (0, L)$ and $\rho_0 > 0$. [4]

(c) Show that with the initial condition as in (b), there exists a critical division rate $\gamma_c(D, L)$ (which you should find explicitly) such that for $\gamma < \gamma_c$ the density decays to zero everywhere but for $\gamma > \gamma_c$ it attains unbounded positive values as $t \to \infty$. [6]

(d) Show that the result in (c) applies for any initial density profile described by a function $\rho(x,0)$ such that $\rho(x,0) \geqslant 0 \; \forall x \in (0,L)$ and $\int_0^L \rho(x,0)\, dx > 0$. [5]

3B Consider the linear second order differential equation

$$Ly(x) = g(x) \quad \text{with} \quad Ly(x) = \frac{d^2}{dx^2}y(x) + 2a\frac{d}{dx}y(x) + (a^2 + b^2)y(x) \, ,$$

for a function $y(x)$ defined on $0 \leqslant x < \infty$. Here a and b are positive constants and g is bounded.

(i) Find the most general solution to $Ly(x) = 0$. [5]

(ii) Consider now the initial conditions

$$y(0) = 0 \, , \qquad \frac{dy}{dx}(0) = 0 \, .$$

Find the Green's function associated to L for these initial conditions. [10]

With this, write the solution to $Ly(x) = g(x)$. [2]

(iii) Assume that $|g(x)| < Ce^{-sx}$ where C and s are positive constants and $s > a$. Prove that $y(x) \to 0$ as $x \to \infty$. [3]

4A The Fourier transform $\tilde{f}(k)$ of a function $f(x)$ may be defined by

$$\tilde{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx\,,$$

where $f(x)$ is sufficiently well behaved for this integral to converge.

(a) Define the inverse of this Fourier transform. [2]

(b) Prove Parseval's theorem, which states that

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\tilde{f}(k)|^2\, dk\,.$$

[You may assume that $\int_{-\infty}^{\infty} e^{imx}\, dx = 2\pi\, \delta(m)$, where $\delta(m)$ is the Dirac δ-function.] [6]

(c) Compute the Fourier transform of the function

$$f(x) = \begin{cases} \sin(x) & \text{when } |x| < \pi/2\,, \\ 0 & \text{otherwise}\,. \end{cases}$$ [6]

(d) Hence show that the integral

$$\int_0^{\infty} \frac{u^2 \cos^2 u}{(u^2 - \frac{\pi^2}{4})^2}\, du = \frac{\pi}{4}\,.$$ [6]

5A A matrix A is said to be *anti-Hermitian* if $A^\dagger = -A$.

(a) Prove that the eigenvalues of an anti-Hermitian matrix are purely imaginary. [2]

(b) Let A be anti-Hermitian and I be the identity matrix. Explain why the matrix $I + A$ is invertible. [2]

Matrices U and V are defined by

$$U = \exp(A) \equiv \sum_{n=0}^{\infty} \frac{A^n}{n!} \qquad \text{and} \qquad V = (I - A)(I + A)^{-1},$$

where A is anti-Hermitian and $(I + A)^{-1}$ is the inverse of $I + A$.

(c) Show that U and V are unitary. [8]

(d) Express the eigenvalues of U and V in terms of the eigenvalues of A. *[You may assume there exists an invertible matrix M such that MAM^{-1} is diagonal.]* [4]

(e) Compute U and V in the case $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. [4]

6A

(a) Prove that eigenvectors of a real symmetric $n \times n$ matrix Q can be chosen to form an orthonormal basis of $\mathbb{R}^n$. [*You may assume without proof that the eigenvalues and eigenvectors of Q are real.*] [4]

(b) For each real number α, let Σ_α be the surface in $\mathbb{R}^3$ given by

$$5x^2 - 3y^2 - 3z^2 + 6xz = \alpha\,.$$

By considering a suitable real symmetric matrix, show there is a new orthonormal basis with associated coordinates (u, v, w) such that Σ_α is given by

$$\lambda u^2 + \mu v^2 + \nu w^2 = \alpha$$

for constants λ, μ, ν which you should determine. [*You are not required to determine the explicit form of the basis change $(x, y, z) \mapsto (u, v, w)$.*] [6]

(c) Sketch the surfaces Σ_1, Σ_0 and Σ_{-1} in the (u, v, w) coordinate system. Use separate axes for each sketch. [6]

(d) Find the (x, y, z) coordinates of the point(s) on Σ_{-1} that are closest to the origin. [4]

7B

(a) (i) Show that

$$\left|\frac{z+ib}{z-ib}\right| = P\ ,$$

where b and P are positive constants, defines a circle in the complex plane. Find the centre and radius of the circle in terms of b and P. [4]

(ii) Consider a real function $V(x,y)$ in two-dimensions in the half-plane $y>0$, which satisfies

$$\nabla^2 V = 0$$

outside a circle of radius R centred on $y=y_0$ and $x=0$, with $y_0>R$. The boundary conditions on the function are

$$V(x,y) = \begin{cases} 0\,, & \text{at} \quad y=0\,, \\ V_0\,, & \text{on the circle}\,. \end{cases}$$

Show that

$$V(x,y) = \frac{V_0}{\cosh^{-1}(y_0/R)} \ln\left|\frac{x+iy+ib}{x+iy-ib}\right|$$

for a suitable value of b that you should determine. [7]

[*Hint: To determine $V(x,y)$, consider looking at the real part of the complex function* $f(z) = \ln\left(\dfrac{z+ib}{z-ib}\right)$.]

(b) Consider the Gamma function, defined via the integral

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt\ ,$$

for $\mathrm{Re}(z) > 0$.

(i) From this definition, show that $\Gamma(1)=1$. [2]

(ii) For $z>0$, using integration by parts, show that

$$\Gamma(z+1) = z\Gamma(z)\,.$$ [3]

(iii) Assuming the identity in part (b.ii) is true for all $z\in\mathbb{C}$, show that $\Gamma(z)$ has poles at all non-positive integers. Determine the order of each pole. [4]

8B Consider the following second order differential equation

$$\frac{d}{dx}\left(x(x-m)\frac{d}{dx}y(x)\right)+\frac{m^3\omega^2}{x-m}y(x)+(x^2+m(x+m))\omega^2 y(x)=0\ , \qquad (\star)$$

where m and ω are real, positive constants.

(a) Identify all singular points of this differential equation and determine whether they are regular or irregular. [3]

(b) Consider a series solution to $(\star)$ around $x=m$ of the form

$$y(x)=\sum_{n=0}^{\infty}a_n\,(x-m)^{n+\nu}\ , \qquad (\dagger)$$

where $a_0\neq 0$. Determine the two candidate values of ν for which a series solution of the form $(\dagger)$ may exist. [4]

(c) Show that the Wronskian of two linearly independent solutions of $(\star)$, up to an overall constant, is of the form

$$W(x)=\exp\left(-\int^x p(u)du\right)\ ,$$

for some function $p(u)$ that you should determine. [3]

(d) Around $x=0$, show that one of the solutions to $(\star)$ can take the form

$$y_1(x)=\sum_{n=0}^{\infty}b_n\,x^n\ ,$$

where $b_0\neq 0$. [3]

Use $W(x)$ in part (c) and y_1 to write an integral expression that gives the other linearly independent solution y_2. [3]

For small values of x, show that

$$y_2(x)=c_0\log(x)+\dots\ ,$$

with c_0 a constant. [4]

9C (a) State the Euler-Lagrange equations resulting from minimization of the integral

$$F = \int_0^T f(x, y, \dot{x}, \dot{y})\, dt, \qquad (*)$$

with respect to a path $(x(t), y(t))$ in the (x, y)-plane whose initial coordinates $(x(0), y(0))$ and final coordinates $(x(T), y(T))$ are fixed. [3]

(b) Show that any term in f of the form $\frac{d}{dt} g(x, y)$ has no effect on these equations. [3]

(c) Now let the path $(x(t), y(t))$ satisfy the constraint equation $\int_0^T u(x, y) dt = 0$. Give the relevant modification to the Euler-Lagrange procedure and briefly describe how any Lagrange multiplier λ is to be determined. [4]

(d) An agricultural vehicle moves across a region of the (x, y) plane of variable terrain, such that the fuel F it consumes obeys $(*)$ with

$$f(x, y, \dot{x}, \dot{y}) = \alpha(1 + \beta x)(1 + \gamma_1 \dot{x} + \gamma_2 \dot{y}) + \frac{\delta_1}{2}\dot{x}^2 + \frac{\delta_2}{2}\dot{y}^2\,,$$

where $\alpha, \beta, \gamma_1, \gamma_2, \delta_1, \delta_2$ are positive constants. The driver of the vehicle seeks a path of fixed duration T from $(x, y) = (0, 0)$ to (S, S) such that F is minimized. Find the Euler-Lagrange equations for this problem. (You are not asked to solve these.) [4]

(e) The driver now decides it would be simpler to instead take the straight-line path for which $x - y = 0$ throughout. The duration T is fixed, as before. Determine $x(t)$ such that F is minimized for this path. [6]

10C Consider the Sturm-Liouville eigenproblem for a self-adjoint operator $\mathcal{L}$:

$$\begin{aligned} \mathcal{L}y_n &= \lambda_n w y_n\,, \\ \text{where}\quad \mathcal{L}y &\equiv -(py')' + qy\,, \end{aligned}$$

for complex functions $y(x)$ of a real variable x in the interval $a \leqslant x \leqslant b$, with real functions $p(x), q(x), w(x)$.

(a) Show that the eigenvalues λ_n are real, and that the eigenfunctions y_n can be chosen real. State without proof the orthonormality relation obeyed by the y_n. [5]

(b) Derive the result (for real y_n)

$$w(\xi)\sum_{n=1}^{\infty} y_n(\xi)y_n(x) = \delta(x-\xi)\,.$$

Derive an expression for the Green's function $G(x;\xi)$ of $\mathcal{L}$ which obeys

$$\mathcal{L}G(x;\xi) = \delta(x-\xi)\,,$$

subject to the same boundary conditions as are obeyed by $y(x)$. [5]

(c) Now take $(a,b) = (0,1)$ with the following boundary conditions on y:

$$y(0) = 0 \quad ; \quad \frac{y'(1)}{y(1)} = \frac{1}{\alpha}\,, \qquad (*)$$

where α is a positive real constant. By considering the case $\mathcal{L}y = -y''$ and $w = 1$, show that any real function $f(x)$ on the interval $0 \leqslant x \leqslant 1$, obeying boundary conditions as in $(*)$, can be written as the "pseudo-Fourier" series

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(q_n x)\,,$$

where q_n is the n-th positive root of $\tan q = \alpha q$. Give an expression for the coefficients a_n. [7]

(d) Using integration by parts confirm explicitly the orthogonality relation

$$\int_0^1 \sin(q_n x)\sin(q_m x)\,dx = 0 \quad \text{for} \quad n \neq m\,,$$

with q_n as defined in part (c). [3]

END OF PAPER