NST0

NATURAL SCIENCES TRIPOS Part IA

Wednesday 12 June 2024 9:00 am to 12:00 pm

MATHEMATICS (2)

Read these instructions carefully before you begin:

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

You may submit answers to **all** of section A, and to no more than **five** questions from section B.

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on one side of the paper only and begin each answer on a separate sheet. (For this purpose, your section A attempts should be considered as one single answer.)

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Complete a gold cover sheet for your section A answer, and place it at the front of your answer to section A.

Complete a gold cover sheet for **each** section B answer, and place it at the front of your answer to that question.

A **separate** green main cover sheet listing all the questions attempted **must** also be completed. (Your section A answer should be recorded just as A: there is no need to list each individual short question.)

Every cover sheet must bear your blind grade number and desk number, and should NOT include your name or CRSid.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
6 gold cover sheets.	No calculators may be used.
Green main cover sheet.	No electronic devices may be used.
Script paper.	
Treasury tag.	

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1

Let $\mathbf{a} = (1, 0, 0)^\top$, $\mathbf{b} = (0, 1, 0)^\top$ and $\mathbf{c} = (0, 1, 1)^\top$.

(a) Find the volume of the parallelepiped formed by these vectors. [1]

(b) Find all the values of the real parameters α, β and γ given that

$$\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} = \mathbf{0}\,. \qquad [1]$$

2

A point has Cartesian coordinates $(x, y, z) = (1, 1, -1)$. For this point, find

(a) the value of r in cylindrical polar coordinates, (r, θ, z); [1]

(b) the value of θ in spherical polar coordinates, (r, θ, ϕ), where angle θ is measured from the positive z-axis. [1]

3

For the function $f(x) = \ln x$ defined on the interval $a \leqslant x \leqslant b$ with $a = 1$ and $b = 2$,

(a) find the value of ξ, such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}\,. \qquad (\dagger) \qquad [1]$$

(b) Sketch the graph of $f(x)$ and add a chord and a tangent to illustrate Eq. ($\dagger$). [1]

4

(a) Let

$$I(x) = \int_a^x f(u)\mathrm{d}u\,,$$

where $f(u)$ is a real-valued continuous function and a is a real constant. Find $\frac{\mathrm{d}I}{\mathrm{d}x}$. [1]

(b) Find

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin x \int_0^x \sin y\,\mathrm{d}y\right]\,. \qquad [1]$$

5

An approximate solution x_1 of the equation $f(x) = 0$ obtained in the first iteration of the Newton-Raphson method is given by the following formula,

$$x_1 = x_0 + \alpha \left(\frac{f'(x_0)}{f(x_0)} \right)^{\beta} ,$$

where x_0 is the initial guess for the solution. Give the expressions for α and β in this formula. [2]

6

For two mutually exclusive events, A and B, state the formulae for the probabilities of the following events:

(a) $P(A \cap B)$, [1]

(b) $P(A \cup B)$. [1]

7

Find the function $f(x, y)$ such that

$$\mathrm{d}f = ye^{xy}\mathrm{d}x + xe^{xy}\mathrm{d}y$$

and $f(0, 0) = 0$. [2]

8

Find the values of

(a) $\exp[\boldsymbol{\nabla}\phi \cdot (\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\phi))]$, [1]

(b) $|\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{F})|^2$, [1]

where ϕ and $\mathbf{F}$ are arbitrary twice-differentiable scalar and vector fields, respectively.

9

Find the matrix elements a_{21} and a_{22} in the following expression:

$$x^2 + 8xy + 4y^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 4 \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} .$$ [2]

10

Without resorting to integration, write down the Fourier series for

$$\sin x \cos x + \cos(\pi/2 - x)\,,$$

where $-\pi \leqslant x < \pi$. [2]

SECTION B

11Z

(a) Let $\hat{\mathbf{u}}$ be a fixed three-dimensional unit vector. Consider the operators $P_{\hat{\mathbf{u}}}$ and $R_{\hat{\mathbf{u}}}$ which can act on any three-dimensional vector as defined below,

$$P_{\hat{\mathbf{u}}}(\mathbf{r}) = \hat{\mathbf{u}}(\mathbf{r} \cdot \hat{\mathbf{u}}),$$
$$R_{\hat{\mathbf{u}}}(\mathbf{r}) = \hat{\mathbf{u}} \times (\mathbf{r} \times \hat{\mathbf{u}}).$$

Give a geometric interpretation of $P_{\hat{\mathbf{u}}}(\mathbf{r})$. [1]

Rewrite $R_{\hat{\mathbf{u}}}(\mathbf{r})$ such that it is expressed using the scalar product, and give a geometric interpretation of $R_{\hat{\mathbf{u}}}(\mathbf{r})$. [3]

(b) Simplify $P_{\hat{\mathbf{u}}}(\mathbf{r}) + R_{\hat{\mathbf{u}}}(\mathbf{r})$. [1]

(c) A straight line passes through points with distinct position vectors $\mathbf{y}$ and $\mathbf{z}$. Show that the position vector $\mathbf{r}$ for points on the line obeys the following equation,

$$\mathbf{r} = \mathbf{p} + t\hat{\mathbf{q}},$$

where t is an arbitrary real parameter. Find expressions for a constant vector $\mathbf{p}$ and a constant unit vector $\hat{\mathbf{q}}$ in terms of $\mathbf{y}$ and $\mathbf{z}$. [2]

(d) (i) Rewrite the equation of the line in part (c) in the form

$$\mathbf{r} \times \hat{\mathbf{a}} = \mathbf{c}$$

for constant unit vector $\hat{\mathbf{a}}$ and constant vector $\mathbf{c}$, relating them to the vectors $\mathbf{p}$ and $\hat{\mathbf{q}}$. [2]

(ii) What geometric property of the line is represented by $\hat{\mathbf{a}}$? [1]

(iii) What geometric property of the line is represented by $c = |\mathbf{c}|$? [2]

(iv) Given $c \neq 0$, complete the following statement "Vector $\mathbf{c}$ is perpendicular to the plane containing ..." [2]

(e) Find the general solution of the vector equation $\mathbf{r} \times \hat{\mathbf{g}} = \mathbf{f}$ and present your answer for $\mathbf{r}$ in terms of the constant unit vector $\hat{\mathbf{g}}$ and constant vector $\mathbf{f}$.

[Hint: What is $R_{\hat{\mathbf{g}}}(\mathbf{r})$?] [6]

12R

Consider the function

$$h(x, y) = x^6 + y^6 - 24x^2y^2 \,.$$

(a) Compute the gradient vector $\boldsymbol{\nabla} h$. [2]

(b) Find all points in the x-y plane where $\boldsymbol{\nabla} h$ is parallel to the x-axis or the y-axis and plot these points in the x-y plane. [4]

(c) By using the results of part (b), or otherwise, find the five stationary points of $h(x, y)$. [4]

(d) Determine the nature of all the stationary points away from the origin. [4]

(e) Analyse the nature of the stationary point at the origin by plotting $h(x, y)$ along the lines $x = 0$, $y = 0$, $y = x$ and $y = -x$ and thus determine if this point is a minimum, or a maximum, or neither. [2]

(f) Make a sketch showing the direction of the gradient vector $\boldsymbol{\nabla} h$ near all stationary points. [4]

13S

(a) Two three-dimensional vector fields $\mathbf{G}$ and $\mathbf{K}$ are given by

$$\mathbf{G} = (\mathbf{a} \cdot \mathbf{r})\mathbf{r}\,, \qquad \mathbf{K} = \frac{\mathbf{r}}{1 + r^2}\,,$$

where $\mathbf{r} = (x, y, z)^\top$ is the position vector, $r = |\mathbf{r}|$, and $\mathbf{a}$ is a fixed non-zero vector. For each of these vector fields, determine whether it can be written as the gradient of a scalar function and, if it can, find all such functions. [8]

(b) A three-dimensional vector field $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} xz + x^2 \\ \mu yz + y^2 \\ \lambda x^2 + y^2 \end{pmatrix},$$

where μ and λ are real parameters.

Compute the line integral $\int \mathbf{M} \cdot \mathrm{d}\mathbf{r}$ from the point $(0, 0, 0)$ to the point $(1, 1, 1)$ using two different paths:

(i) A straight line from the point $(0, 0, 0)$ to the point $(1, 1, 1)$. [3]

(ii) From the point $(0, 0, 0)$ to the point $(1, 1, 0)$ along the shortest arc of the unit circle in the (x, y)-plane centered on $(1, 0, 0)$, and then along the straight line from the point $(1, 1, 0)$ to the point $(1, 1, 1)$. [6]

Determine the values of μ and λ for which the vector field $\mathbf{M}$ is conservative and check that they agree with the results of parts (i) and (ii). [3]

14X

A training run is timed using a digital watch which may be assumed to keep accurate time but the display is truncated to show only hours and minutes, for example 09h00m55s would display as 09:00. At the start of the run the watch does indeed read 09:00.

(a) Let the continuous random variable S be the difference between the actual start time and 09h00m00s. Let the uniform probability density function of S be $p(s)$.

(i) Express $p(s)$ algebraically. [2]

(ii) Sketch $p(s)$. [1]

(iii) Calculate the mean and variance for the distribution. [2]

(b) At the end of the run the display reads 09:25. Consider the actual time taken for the run as a continuous random variable and let T be the difference between the actual time taken and 25 minutes.

(i) Show that the probability density function for T is $q(t)$, where

$$q(t) = \begin{cases} 0 & \text{for } \; t < -1\,, \\ 1+t & \text{for } \; -1 \leqslant t < 0\,, \\ 1-t & \text{for } \; 0 \leqslant t < 1\,, \\ 0 & \text{for } \; t \geqslant 1\,. \end{cases}$$

[5]

(ii) Sketch $q(t)$ and show that it is normalised. [2]

(iii) Calculate the mean and standard deviation for T. [2]

(iv) What is the probability that the run took more than 25m30s? [2]

(c) For the sum of independent random variables statistical theory predicts: the mean is the sum of the means; the variance is the sum of the variances. Check and explain why your results for part (b) are consistent or inconsistent with these predictions. [4]

15V

(a) Consider the second order differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \beta\frac{\mathrm{d}y}{\mathrm{d}x} + \gamma y = 0\,,$$

where β and γ are real constants.

(i) When $\beta = 4$, find γ such that the general solution is $y(x) = Axe^{-2x} + Be^{-2x}$, where A and B are arbitrary constants. [3]

(ii) Find all the solutions for $y(x)$ when $\beta = 6$ and $\gamma = 5$. [4]

(iii) Find all the solutions for $y(x)$ when $\beta = 3$ and $\gamma = 4$. [5]

(b) Consider the second order differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = 125x^2\,.$$

Find the particular solution subject to boundary conditions

$$y(0) = 1$$

and

$$y\left(\frac{\pi}{2}\right) = \frac{25\pi^2}{4} + 20\pi + 22\,.$$ [8]

16W

(a) Find $\nabla^2\phi$ for $\phi(x, y, z) = e^{x^2yz}$. [4]

(b) The vertices of the tetrahedron OABC have the following coordinates, $O(0,0,0)$, $A(1,0,0)$, $B(0,1,0)$, and $C(0,0,1)$. Find the outward flux of the vector field

$$\mathbf{F} = \begin{pmatrix} x+y+z \\ (x+y+z)^2 \\ 1-x-y-z \end{pmatrix}$$

through

(i) the face OBA and thus demonstrate that it is equal to $-\frac{1}{6}$; [4]

(ii) each of the other three faces of the tetrahedron; [11]

(iii) the closed surface of the tetrahedron. [1]

17Y

Consider the following set of simultaneous equations, in which α and β are real constants,

$$\begin{cases} x + 2\alpha y + \beta z = 1\,, \\ x - y + 3z = -1\,, \\ -3x + 3y - 3\alpha z = 3\,. \end{cases}$$

(a) Express this set of linear equations in matrix form $\mathbf{A}\mathbf{x} = \mathbf{b}$. [1]

(b) Calculate $\det(\mathbf{A})$ as a function of α and β. For which values of α and β does this determinant equal zero? [4]

(c) Solve this set of simultaneous equations for

(i) $\alpha \neq 3$ and $\alpha \neq -\frac{1}{2}$, [5]

(ii) $\alpha = 3$, [4]

(iii) $\alpha = -\frac{1}{2}$. [1]

(d) Give a geometric interpretation for all cases considered in part (c) by specifying the relative positions of the planes (such as being parallel, coinciding, intersecting) described by each of the three simultaneous equations. Your answer should distinguish between different values of β. [5]

18T

(a) Consider the following set of functions:

$$\{1,\ \cos(\pi x/L),\ \sin(\pi x/L),\ \cos(2\pi x/L),\ \sin(2\pi x/L),\ \cos(3\pi x/L),\ \sin(3\pi x/L),\ \ldots\}\,.$$

Explain what it means to say that they are *orthogonal* on the interval $-L \leqslant x < L$, where L is a positive constant. [2]

[For the rest of this question you may assume that these functions are orthogonal on this interval.]

(b) A real function $f(x)$ is periodic with period $2L$ and has Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]\,.$$

(i) Show that $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\pi x/L)\mathrm{d}x$. [2]

(ii) Write down a corresponding expression for a_n. [1]

(iii) Prove Parseval's identity

$$\frac{1}{2L}\int_{-L}^{L} f(x)^2\mathrm{d}x = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)\,. \qquad [4]$$

(c) A function $f(x)$ is periodic with period 2 and $f(x) = x(1-|x|)$ for $-1 \leqslant x < 1$.

(i) Determine the Fourier series of $f(x)$. [7]

(ii) Hence show that

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^6} = A\pi^6\,,$$

where A is a fraction that you should determine. [4]

19Y*

(a) Consider a cylinder of height h and base radius r. The cylinder is inscribed within a sphere of fixed radius R such that the perimeters of its two end-discs lie on the surface of the sphere. Using the method of Lagrange multipliers, determine the values of r and h that maximise the volume of the cylinder. Calculate the maximum volume of the cylinder as a fraction of the volume of the sphere. [8]

(b) Consider an ellipse defined by

$$4x^2 + \frac{y^2}{16} = 1\,.$$

(i) Using the method of Lagrange multipliers, find all points (x, y) lying on the ellipse that extremise the function $f(x, y) = xy$ and determine the value of f at each of these points. [9]

(ii) Give a geometric interpretation of the maximum value of f found in part (i). [3]

20Z*

Consider the one-dimensional wave-equation

$$\frac{\partial^2\psi}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} = 0$$

for $0 < x < \pi$ and $t > 0$, where v is a real constant, and boundary conditions for $t > 0$ are given by

$$\frac{\partial\psi}{\partial x}(0, t) = 0\,,$$
$$\frac{\partial\psi}{\partial x}(\pi, t) = 0\,.$$

(a) Explain the principle of superposition for solutions to the wave equation. [2]

(b) We now impose the following initial conditions for $0 < x < \pi$:

$$\psi(x, 0) = \cos^2\frac{x}{2}\,,$$
$$\frac{\partial\psi}{\partial t}(x, 0) = 0\,.$$

By using the method of separation of variables, find the solution of the wave equation $\psi(x, t)$, obeying the initial and boundary conditions specified above. [10]

(c) Now consider the wave equation but with the spatial domain extended to the entire real line, for $t > 0$ and subject to

$$\psi(x, 0) = \cos^2\frac{x}{2},$$
$$\frac{\partial\psi}{\partial t}(x, 0) = 0,$$

where $-\infty < x < \infty$.

(i) By looking for a solution of the form

$$\psi(x, t) = f(x - vt) + g(x + vt)\,,$$

where f and g are some functions, find $\psi(x, t)$, and compare it with the solution obtained in part (b). [6]

(ii) Sketch graphs of $\psi(x, t)$ found in part (i) versus x for $t = 0$ and for several other moments of time (which you should specify) so that your graphs illustrate the time evolution of $\psi(x, t)$. [2]

END OF PAPER