UNIVERSITY OF CAMBRIDGE

CST0/MATH/1

NST0

NATURAL SCIENCES TRIPOS Part IA

Monday 10 June 2024 9:00 am to 12:00 pm

MATHEMATICS (1)

Read these instructions carefully before you begin:

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

You may submit answers to **all** of section A, and to no more than **five** questions from section B.

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on one side of the paper only and begin each answer on a separate sheet. (For this purpose, your section A attempts should be considered as one single answer.)

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Complete a gold cover sheet for your section A answer, and place it at the front of your answer to section A.

Complete a gold cover sheet for **each** section B answer, and place it at the front of your answer to that question.

A **separate** green main cover sheet listing all the questions attempted **must** also be completed. (Your section A answer should be recorded just as A: there is no need to list each individual short question.)

Every cover sheet must bear your blind grade number and desk number, and should NOT include your name or CRSid.

Tie up your answers and cover sheets into **a single bundle**, with the main cover sheet on top, and then the cover sheet and answer for each question, in the numerical order of the questions.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
6 gold cover sheets.	No calculators may be used.
Green main cover sheet.	No electronic devices may be used.
Script paper.	
Treasury tag.	

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1

(a) Calculate the modulus of the vector $(1, 2, 1)^\top$. [1]

(b) Calculate the angle between the vectors $(1, 2, -1)^\top$ and $(0, -1, 1)^\top$. [1]

2

Find the gradient and y-intercept for the tangent to the curve

$$y = x^2 + \ln x$$

at the point where $x = 1$. [2]

3

Differentiate $f(x) = \exp(2 - x^2)$ with respect to x and find the maximum value of $f(x)$. [2]

4

(a) Calculate the sum of the odd integers from 5 to 55 inclusive. [1]

(b) Calculate the sum of the infinite geometric series with first term 4 and common ratio $\frac{1}{3}$. [1]

5

Solve the following inequality for x:

$$x \geqslant (x-1)^2.$$ [2]

6

(a) How many real solutions exist for the following simultaneous equations?

$$\begin{cases} x^2 + y^2 = 1\,, \\ y = (x-1)^2\,. \end{cases}$$ [1]

(b) What are they? [1]

7

Simplify the following equation

$$2\sec x - \tan x = \sqrt{3}$$

and then solve it for x in the range $0 \leqslant x < \pi/2$. [2]

8

Find the radius and the coordinates of the centre of the following circle:

$$4x^2 - 8x + 4y^2 + 4y = 11\,.$$ [2]

9

Find the values of a and n such that, for sufficiently small nonzero $|x|$,

$$(1 + ax)^n = 1 - 2x + 3x^2 + \dots\,.$$ [2]

10

(a) Calculate the indefinite integral of $x\cos x$. [1]

(b) Calculate the area between the x-axis and the curve $y = \sin x$ in the interval $0 \leqslant x \leqslant \pi$. [1]

SECTION B

11Z

(a) Show that for any two complex numbers, z_1 and z_2,

(i) $(z_1 + z_2)^* = z_1^* + z_2^*$, [1]

(ii) $(z_1 z_2)^* = z_1^* z_2^*$, [1]

where z^* is the complex conjugate of z.

(b) By considering the complex numbers $z_1 = 1 + iA$ and $z_2 = 1 + iB$, where A and B are real, derive the arctangent addition formula:

$$\arctan(A) + \arctan(B) = \arctan\left(\frac{A+B}{1-AB}\right) .$$

[You may assume that the arguments of all complex numbers used in the derivation lie between $-\pi/2$ and $\pi/2$.] [2]

(c) Show that for any two complex numbers z_1 and z_2, the quantity $Q = z_1 z_2^*$ does not change if both z_1 and z_2 are rotated by an angle α around the origin of the complex plane. [2]

(d) Treating z_1 and z_2 as vectors in the complex plane, give an interpretation of

(i) the real part of $Q = z_1 z_2^*$, [1]

(ii) the imaginary part of $Q = z_1 z_2^*$. [1]

(e) Show that for real x and y obeying $x = \cos y$, the following relation holds:

$$y = \pm i \ln\left(x + i\sqrt{1-x^2}\right) + 2n\pi ,$$

for integer n. [6]

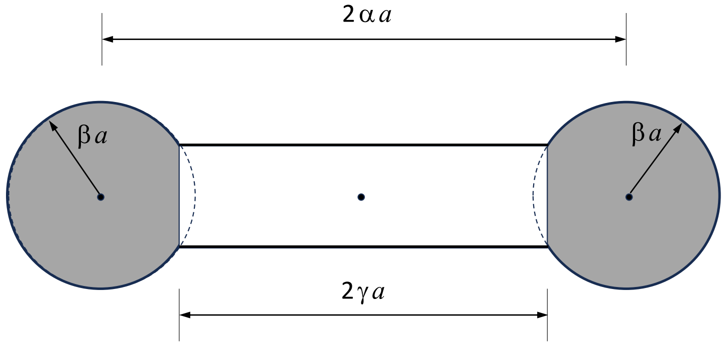
(f) For $\theta \neq 2p\pi$, where p is an integer, show that

$$\sum_{n=0}^{N-1} \cos n\theta = \frac{\cos\frac{(N-1)\theta}{2} \sin\frac{N\theta}{2}}{\sin\frac{\theta}{2}} .$$

What value does the sum take when $\theta = 2p\pi$? [6]

12S

An axisymmetric dumbbell (see figure) consists of a cylindrical handle and two identical end pieces (grey) shaped as truncated spheres with the caps removed for smooth attachment to the handle as shown in the figure. The truncated spheres have radii βa and their centres are distance αa from the centre of the dumbbell. The length of the handle is $2\gamma a$. The positive constants α, β and γ satisfy $\alpha - \beta < \gamma < \alpha$, and the constant a has the dimension of length. The dumbbell is made of material of uniform density ρ.



(a) Show that the radius of a circular cross-section of the cylindrical handle is

$$a\sqrt{\beta^2 - (\alpha - \gamma)^2}.$$ [1]

(b) Calculate the mass of the handle using multiple integration in cylindrical polar coordinates. [3]

(c) Calculate the mass of each end piece (a truncated sphere) using multiple integration in suitably chosen cylindrical polar coordinates. [7]

(d) Repeat the calculation in part (c) using multiple integration in suitably chosen spherical polar coordinates. [8]

(e) Calculate the total mass of the dumbbell. [1]

13Y

(a) Find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4y^2}{x^2} - x^2y^2\,. \qquad [6]$$

Determine the solution in each of the following cases:

(i) $y(1) = 1\,,$ [1]

(ii) $y(2) = 0\,.$ [1]

(b) Consider the following differential form,

$$\mu(x)\left(xy - 16x - x^3\right)\mathrm{d}x + \mu(x)\left(16 + x^2\right)\mathrm{d}y\,,$$

where $\mu(x)$ is an unknown real-valued differentiable function.

(i) Find a function $\mu(x)$ for which this differential form is exact. [6]

(ii) Hence or otherwise, find in explicit form the general solution of the equation

$$\left(xy - 16x - x^3\right) + \left(16 + x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0\,. \qquad [6]$$

14R

(a) A right-angled triangle of area A has sides of length a, b and c, where c is the hypotenuse. Small changes $\mathrm{d}a$ and $\mathrm{d}b$ are made to the sides a and b, respectively. Find expressions for the fractional change of the hypotenuse $\mathrm{d}c/c$ and the fractional change in the area of the triangle $\mathrm{d}A/A$ in terms of a, b and their changes. Find the fractional change in area if a increases by 1% and b decreases by 2%. [6]

(b) Consider a function $z(x, y)$ defined implicitly by the equation

$$x - \alpha z = \varphi(y - \beta z),$$

where α and β are real constants and φ is an arbitrary differentiable function. Show that $z(x, y)$ satisfies

$$\alpha \frac{\partial z}{\partial x} + \beta \frac{\partial z}{\partial y} = 1\,.$$ [6]

(c) Consider a function $z(x, y)$ that satisfies $z(\lambda x, \lambda y) = \lambda^n z(x, y)$ for an arbitrary real positive λ and an integer n. Show that

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = nz$$

and

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = n(n-1)z\,.$$ [8]

15T

(a) State Taylor's theorem by giving the series expansion about $x = a$ of a function $f(x)$ that is $(n + 1)$-times differentiable, showing the first $n + 1$ terms, together with an expression for the remainder term R_{n+1}. [4]

(b) Taking $f(x) = \ln x$ and $n = 2$, use the Taylor series expansion of $f(x)$ about $x = 1$ to obtain an approximate value for $\ln(1.1)$. Explain whether your answer is an overestimate or an underestimate, and give an upper bound on the error. [4]

(c) Using any method, obtain the first 3 non-zero terms in the Taylor series expansions of

(i) $(\cosh x)^{-1/2}$ about $x = 0$; [6]

(ii) $e^{\sin x}$ about $x = \pi/2$. [6]

[You may quote and use standard Taylor series expansions of known functions.]

16W

A box contains white and black balls. Initially, a fraction x of the balls are white and the remaining balls are black. An integer number of balls, $k > 0$, is taken from the box without replacement. Let p_{w} be the probability that these balls taken from the box are all white.

(a) Consider a particular case of $x = \frac{1}{2}$ and $k = 2$.

(i) Show that the total number of balls in the box is $N = \frac{2(1-2p_{\mathrm{w}})}{1-4p_{\mathrm{w}}}$. [4]

(ii) Find in terms of p_{w} the probability, p_{d}, that the balls taken from the box are of different colours. [2]

(b) Consider a more general case of $0 \leqslant x \leqslant 1$ and $k = 2$.

(i) Find p_{d} in terms of x and p_{w}. [3]

(ii) Given that both balls taken from the box are of the same colour, find the probability that they are black. Express your answer in terms of x and p_{w}. [3]

(c) Let $x = \frac{1}{2}$.

(i) For $k = 3$, it is given that the probability p_{w} that all three balls are white is $p_{\mathrm{w}} = \frac{2}{19}$. Find N, the total number of balls in the box. [3]

(ii) From the box containing N balls, with N found in part c(i), $k = 12$ balls are taken without replacement. Find the probability that 4 of them are white. Express your answer in the form of a fraction,

$$\frac{\binom{N_1}{m_1}\binom{N_2}{m_2}}{\binom{N}{k}},$$

where you need to find the integers N_1, N_2, m_1 and m_2. [5]

17R

Evaluate the following integrals:

(a)

$$\int_4^9 \frac{\mathrm{d}x}{\sqrt{x}-1}\,; \qquad [4]$$

(b)

$$\int_{\pi/4}^{\pi/3} \frac{1+\tan^2 x}{(1+\tan x)^2}\mathrm{d}x\,; \qquad [5]$$

(c)

$$\int \frac{e^{2x}-2e^{x}}{e^{2x}+1}\mathrm{d}x\,; \qquad [4]$$

(d)

$$\int \frac{\mathrm{d}x}{1+3\cos^2 x}\,. \qquad [7]$$

18V

(a) Let $\mathbf{A} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 4 & 1 \end{pmatrix}$.

(i) Compute the eigenvalues and normalised eigenvectors of $\mathbf{A}$. [9]

(ii) Compute the inverse of $\mathbf{B}$ and hence find all the solutions $\mathbf{x}$ of the matrix equation

$$\mathbf{B}\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$ [5]

(b) A 3×3 matrix $\mathbf{M}$ represents a linear transformation and has eigenvalues 1, 1, and -1. The eigenvectors are $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ but they do not necessarily correspond to the eigenvalues in that order.

(i) Give a geometric interpretation of all the transformations that $\mathbf{M}$ could represent. [4]

(ii) Calculate $\mathbf{M}^2 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. [2]

19T*

(a) Determine the 100^{th} derivative of $x^2 e^x$. [3]

(b) (i) State the ϵ, δ definition of what it means for a function $f(x)$ to be continuous at $x = a$. [2]

(ii) Let p be a real number and let

$$f(x) = \begin{cases} |x|^p \cos(1/x) & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0\,. \end{cases}$$

For what values of p is $f(x)$ continuous at $x = 0$? For those values, give a proof of continuity at $x = 0$ using the ϵ, δ definition. [5]

(iii) For what values of p is $f(x)$ differentiable at $x = 0$? What is $f'(0)$ for those values of p? [3]

(c) Determine whether the following series are convergent:

(i)

$$\sum_{n=1}^{\infty} \frac{n^4}{3^n}\,; \qquad [3]$$

(ii)

$$\sum_{n=1}^{\infty} \frac{n^2 + 2n}{n^3 + 3n^2 + 1}\,. \qquad [4]$$

Give brief justification of your answers.

[You may assume standard results for the convergence of series of the form $\sum_{n=1}^{\infty} n^{-p}$.]

20V*

(a) Write down the formula for

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_{a(x)}^{b(x)} f(x,t)\mathrm{d}t\right],$$

assuming that all relevant functions are differentiable and integrable. The derivation is not required. [2]

(b) The function $h(x)$ is defined by

$$h(x) = \int_{x}^{\pi x^2} \left(2x^2 t + x\sin t\right)\mathrm{d}t\,.$$

Find $\frac{\mathrm{d}h}{\mathrm{d}x}$ by

(i) using the formula quoted in part (a), [5]

(ii) evaluating the integral and then differentiating with respect to x. [3]

(c) Let

$$g(x) = \exp\left(\int_0^1 \frac{t^x - 1}{\ln t}\mathrm{d}t\right).$$

(i) Derive a first-order differential equation for $g(x)$. [4]

(ii) Specify the range of x for which this equation is valid. [2]

(iii) Determine $g(0)$ and use this as a boundary condition to solve the differential equation derived in part c(i). [2]

(iv) Find $\frac{\mathrm{d}^3}{\mathrm{d}x^3}\left[\int_x^{-1}\left(\int_{x_1}^{-1} g(x_2)\mathrm{d}x_2\right)\mathrm{d}x_1\right]$. [2]

END OF PAPER