UNIVERSITY OF CAMBRIDGE

NST2PC/MTH/2

NST1

NATURAL SCIENCES TRIPOS Part IB

Saturday, 10 June, 2023 9:00am to 12:00pm

MATHEMATICS (2)

Read these instructions carefully before you begin:

You may submit answers to no more than **six** questions. All questions carry the same number of marks.

The approximate number of marks allocated to a part of a question is indicated in the right-hand margin.

Write on **one** side of the paper only and begin each answer on a separate sheet.

At the end of the examination:

Each question has a number and a letter (for example, **8C**).

Tie up **each** answer in a **separate** bundle, marked with the question number.

Do not join the bundles together.

For each bundle, a gold cover sheet **must** be completed and attached to the bundle.

A **separate** green master cover sheet listing all the questions attempted **must** also be completed.

Every cover sheet must bear your examination number and desk number.

Calculators and other electronic or communication devices are not permitted in this examination.

STATIONERY REQUIREMENTS
6 gold cover sheets and treasury tags
Green master cover sheet
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1B

(a) The inner product of two functions f and g defined on $[a, b]$, with respect to a weight $w(x) > 0$, is defined by

$$\langle f|g\rangle = \int_a^b f^*(x)g(x)w(x)\,dx\,. \qquad (*)$$

Consider the operator

$$\mathcal{L} = -\frac{1}{w(x)}\left[\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) - q(x)\right], \qquad x \in [a, b]\,,$$

where $p(x)$ and $q(x)$ are real functions with $p(x) > 0$.

(i) Derive the boundary condition under which $\mathcal{L}$ is self-adjoint with respect to the inner product defined in $(*)$. [6]

(ii) Show that any two eigenfunctions of $\mathcal{L}$ with distinct eigenvalues are orthogonal with respect to the inner product defined in $(*)$. [2]

(b) Consider the eigenvalue problem

$$\mathcal{L}y = \lambda y\,, \quad \text{where} \quad \mathcal{L}y = -x^2y'' - xy' - y\,, \qquad (\dagger)$$

with boundary conditions $y(1) = y(e) = 0$.

(i) Rewrite $\mathcal{L}$ as defined in $(\dagger)$ in Sturm-Liouville form, and identify the corresponding functions $p(x)$, $q(x)$ and $w(x)$. [2]

(ii) Find the eigenvalues λ_n and orthonormal eigenfunctions y_n of $\mathcal{L}$. [6]

(iii) Determine the solution to the equation $\mathcal{L}y = 1$ as an eigenfunction expansion. [4]

2B

In plane polar coordinates (r, θ), the Laplace equation has the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = 0\,. \qquad (*)$$

(a) Use separation of variables to show that the general solution of $(*)$ for $r > 0$ can be written as

$$\Phi(r,\theta) = A_0 + B_0 \ln r + \sum_{n=1}^{\infty} \left[\left(A_n r^n + B_n r^{-n}\right)\cos n\theta + \left(C_n r^n + D_n r^{-n}\right)\sin n\theta\right]\,,$$

where A_n, B_n, C_n and D_n are constants. [10]

(b) The surface of an infinite cylinder is given by $r = R$ in cylindrical polar coordinates (r, θ, z). The cylinder has a surface charge density $\sigma(\theta)/4\pi$ so the electrostatic potential Φ is continuous at $r = R$, but its normal derivative has a discontinuity

$$\left.\frac{\partial \Phi}{\partial r}\right|_{r=R^+} - \left.\frac{\partial \Phi}{\partial r}\right|_{r=R^-} = -\sigma(\theta)\,,$$

where the values at $r = R^+$ and $r = R^-$ are the limits as $r \to R$ from above and below, respectively. Suppose the function $\sigma(\theta)$ can be expanded as the Fourier series

$$\sigma(\theta) = \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)\,.$$

Assume that Φ is independent of z and therefore satisfies $(*)$ for $r < R$ and $r > R$. Determine Φ for all r, assuming that $\Phi \to 0$ as $r \to \infty$ and that Φ is finite as $r \to 0$. [10]

3C

(a) For smooth functions Φ and Ψ defined in a volume V bounded by a closed surface S, prove *Green's identity*,

$$\int_V (\Phi\nabla^2\Psi - \Psi\nabla^2\Phi)\, dV = \oint_S \left(\Phi\frac{\partial\Psi}{\partial\boldsymbol{n}} - \Psi\frac{\partial\Phi}{\partial\boldsymbol{n}}\right) dS\,,$$

where $\boldsymbol{n}$ is the unit vector normal to S. Use this identity to derive a general solution to the Laplace equation with Dirichlet boundary conditions

$$\begin{aligned} \nabla^2\Phi &= 0 \quad &&\text{for } \boldsymbol{r} \in V\,, \\ \Phi &= f(\boldsymbol{r}) \quad &&\text{for } \boldsymbol{r} \in S\,. \end{aligned}$$

[Express your solution in terms of the Green's function that vanishes on the boundary.]

[8]

(b) The Green's function G for the Laplacian ∇^2 in unbounded three-dimensional space is

$$G(\boldsymbol{r}, \boldsymbol{r}') = -\frac{1}{4\pi}\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}\,.$$

(i) Show that $G(\boldsymbol{r}, \boldsymbol{r}')$ satisfies the Laplace equation for all $\boldsymbol{r}' \neq \boldsymbol{r}$. [2]

(ii) For $r' = |\boldsymbol{r}'| < a$, show that the function

$$H(\boldsymbol{r}, \boldsymbol{r}') = G(\boldsymbol{r}, \boldsymbol{r}') - \frac{a}{r'}G(\boldsymbol{r}, \boldsymbol{r}'')\,,$$

where $\boldsymbol{r}'' = \dfrac{a^2}{r'^2}\boldsymbol{r}'$, vanishes on the sphere $|\boldsymbol{r}| = a$. Comment on the limit $r' \to 0$. [3]

(c) For the Laplacian ∇^2, find the Green's function, which vanishes on the boundaries, for the following regions:

(i) The interior of the half sphere $r < a$ and $\theta < \pi/2$, where (r, θ, ϕ) are spherical polar coordinates; [3]

(ii) The region $-1 < x < 1$, where (x, y, z) are Cartesian coordinates. [4]

4A

(a) State the conditions that must be satisfied for a general function $g(z)$ to be analytic at the point $z = z_0 \in \mathbb{C}$. [2]

(b) Consider the function

$$f(z) = \frac{1 + z + z^2}{(1 + 2z^2 + z^4)\, z^{1/2}} \, .$$

Identify points in the complex plane where $f(z)$ is not analytic, stating the nature of $f(z)$ at these points. [4]

(c) For the integral

$$I = \int_0^\infty f(z)\, dz \, ,$$

identify a contour Γ suitable for integrating around and establish how the integrals along the different parts of the contour relate to I. [6]

(d) Evaluate the integral

$$C = \oint_\Gamma f(z)\, dz$$

and hence determine I. [8]

5A

A moving body emitting a signal passes a receiver at time $t = 0$. Due to Doppler shifting, the signal reaching the receiver is given by

$$f(t) = \begin{cases} e^{\lambda t} \cos(2t) \, , & t < 0 \, , \\ e^{-\lambda t} \cos(t) \, , & t \geqslant 0 \, , \end{cases}$$

for some real constant $\lambda > 0$. Due to linear damping, the response $y(t)$ of the receiver obeys

$$\ddot{y} - y = -f(t) \, .$$

(a) Sketch $f(t)$ and compute its Fourier transform $\widetilde{f}(\omega)$. [7]

(b) For the case $\lambda = 1$, determine $\widetilde{y}(\omega)$, the Fourier transform of $y(t)$ and hence find $y(t)$ for $t < 0$. [13]

6C

(a) Consider the rank-4 tensor defined by contracting two rank-3 alternating (Levi–Civita) symbols together:

$$T_{ijkl} = \epsilon_{ijm}\epsilon_{klm} \, .$$

(i) Show that T_{ijkl} is isotropic and transforms as a tensor. [2]

(ii) The most general isotropic rank-4 tensor is

$$\alpha \, \delta_{ij}\delta_{kl} + \beta \, \delta_{ik}\delta_{jl} + \gamma \, \delta_{il}\delta_{jk} \, , \qquad (*)$$

where δ_{ij} is the Kronecker delta. Determine the specific values of α, β, γ in $(*)$ for T_{ijkl}. [4]

(b) Consider the integral

$$I_{ijkl} = \int\!\!\int\!\!\int_B x_i x_j x_k x_l \; dx_1 \, dx_2 \, dx_3 \, ,$$

where $\boldsymbol{x} = (x_1, x_2, x_3)$ is the position vector in Cartesian coordinates and B is the unit solid sphere defined by $|\boldsymbol{x}| \leqslant 1$.

(i) Explain why the tensor I_{ijkl} must be isotropic, and determine the values of its α, β, γ coefficients in $(*)$ up to an overall multiplicative coefficient. [2]

(ii) Evaluate the special case I_{iijj} (summing over repeated indices). [2]

(iii) Using your answers in the previous parts, or otherwise, evaluate the integral I_{ijkl}. [4]

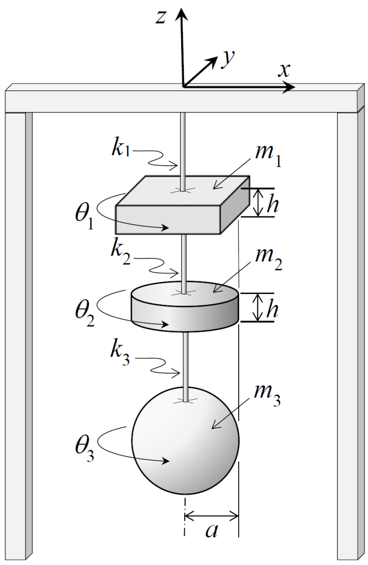
(c) The matrix M_{ij} can be considered as a rank-2 tensor. Provided M_{ij} is not singular, the matrix inverse $(M^{-1})_{ij}$ is the rank-2 tensor such that $M_{ij}(M^{-1})_{jk} = \delta_{ik}$. Let $Y_{ij} = (B^{-1})_{ij}$ where

$$B_{ij} = \delta_{ij} + \lambda X_{ij} \, .$$

Using tensor notation, calculate Y_{ij} as an expansion in λ, neglecting $O(\lambda^4)$ and higher terms. [Note that X_{ij} is not necessarily symmetric.] [6]

7A

An artist decides to create an art installation comprising three objects able to rotate about the vertical z axis within a rigid frame, as shown in the figure below.



Looking from above, the top object is square with sides $2a$, uniform thickness h and mass m_1. The middle object is a disk of radius a with thickness h and mass m_2. The bottom object is a sphere of radius a with mass m_3. All three objects are solid but their densities ρ_i differ. The rotation of the objects is resisted by tortional springs such that the torque applied to an object by spring j varies as $|\tau_j| = k_j|\Delta\theta_j|$, where k_j are the spring constants and $\Delta\theta_j$ is the difference in rotation between the ends of the spring.

(a) The moment of inertia of a body rotating about its z axis is given by

$$I = \int_V (x^2 + y^2)\rho\, dV\,.$$

Show that the moments of inertia of the three masses are given by $I_1 = \frac{1}{6}m_1a^2$, $I_2 = \frac{1}{2}m_2a^2$ and $I_3 = \frac{2}{5}m_3a^2$. [4]

(b) Give expressions for the rotational kinetic energy of the objects and the elastic energy stored in the tortional springs. Hence, determine the Lagrangian for the installation. Determine also the corresponding Euler-Lagrange equations. [*Hint: The elastic energy of a tortional spring is determined from* $\int \tau d\theta$.] [5]

(c) Take $a = 1$, $m_1 = 3$, $m_2 = 1/2$, $m_3 = 1/5$, $k_1 = 3$, $k_2 = 2$ and $k_3 = 1$. Show that $\omega_1 = \sqrt{6}$ is a normal frequency of the installation and determine the corresponding normal mode. Find the other normal frequencies ω_2 and ω_3. [11]

8C

(a) For each of the four algebras below, either show that it is not a group (with respect to the operation $*$), or else verify that it satisfies the group axioms:

(i) $\mathbb{R}$ under the operation $a * b = a - b$.

(ii) $\mathbb{C} \setminus \{0\}$ under the operation $a * b = ab/(a+b)$.
[The notation $A \setminus B$ means the elements of set A excluding those in set B.]

(iii) $\mathbb{R} \cup \{-\infty\}$ under the operation $a * b = \log(e^a + e^b)$, where we stipulate as definitions that $e^{-\infty} = 0$ and hence $\log(0) = -\infty$.

(iv) The set of odd integers, under the operation $a * b = a + b + 1$. [5]

(b) Explain what is meant by *left coset* and *right coset*, and a *normal subgroup*. Show that for a normal subgroup, the left and right cosets are identical. [3]

(c) Given a finite group G of order $|G|$ and a normal subgroup H of order $|H|$, define the *quotient group* G/H and verify that it is indeed a group. What does Lagrange's theorem say about $|G|$ and $|H|$? Supposing that a group G is of prime order p, prove that it is abelian. [7]

(d) Consider the set S of 2×2 matrices of the form

$$\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix},$$

where $a, b \in \mathbb{C}$, and * is complex conjugation. Show that S is not a group under matrix multiplication. Show also that by the removal of a single element $z \in S$, the remaining elements $S \setminus \{z\}$ form a group. [5]

9C

(a) Consider the group G of rotational symmetries of a regular tetrahedron (excluding reflections). What is the order of this group? Identify the conjugacy classes and give the number of elements in each class. [8]

(b) Identify a homomorphism Φ mapping the tetrahedron group G onto the cyclic group with 3 elements. What is the kernel K of Φ? Verify that K is a normal subgroup. [7]

(c) Consider the group $G(p,q)$ consisting of ordered pairs (n,m), where $0 \leqslant n < p$ and $0 \leqslant m < q$, and p,q,n,m are all integers. The group operation is

$$(n,m) * (n',m') = \big((n+n') \bmod p,\ (m+m') \bmod q\big)\,.$$

[The modulo operation $k \bmod r = s$ means that s is the smallest non-negative integer such that $k-s$ is divisible by r.]

Prove that if p and q are coprime (i.e. their only common factor is 1), then $G(p,q)$ is isomorphic to C_{pq}, the cyclic group of order pq. Show that if p and q are not coprime, then $G(p,q)$ is not cyclic. [5]

10A

The group $G = \{g_0, g_1, \ldots, g_7\}$ (where $g_0 = I$ is the identity) has generators $g_1 = U$ and $g_4 = V$ with the three-dimensional real representations

$$U = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad V = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

(a) Show that $U^4 = I$, $V^2 = I$ and $U^pV = VU^q$ where $q = 4 - (p \bmod 4)$. [The modulo operation $k \bmod r = s$ means that s is the smallest non-negative integer such that $k-s$ is divisible by r.] [*Hint: Rewrite U as a rotation matrix.*] [3]

(b) Show that G is a group under matrix multiplication and determine the group table. [*Hint: Check which products are commutative.*] [5]

(c) What is the character of this three-dimensional representation? [2]

(d) Identify all possible subgroups $G_1, G_2, \ldots$ of G and their order. Identify any subgroups that are isomorphic. [5]

(e) Determine two-dimensional and one-dimensional faithful representations of any cyclic subgroups identified in part (d). [5]

END OF PAPER