UNIVERSITY OF CAMBRIDGE

NST2PC/MTH/1

NST1

NATURAL SCIENCES TRIPOS Part IB

Wednesday, 31 May, 2023 9:00am to 12:00pm

MATHEMATICS (1)

Read these instructions carefully before you begin:

You may submit answers to no more than **six** questions. All questions carry the same number of marks.

The approximate number of marks allocated to a part of a question is indicated in the right-hand margin.

Write on **one** side of the paper only and begin each answer on a separate sheet.

At the end of the examination:

Each question has a number and a letter (for example, **8C**).

Tie up **each** answer in a **separate** bundle, marked with the question number.

Do not join the bundles together.

For each bundle, a gold cover sheet **must** be completed and attached to the bundle.

A **separate** green master cover sheet listing all the questions attempted **must** also be completed.

Every cover sheet must bear your examination number and desk number.

Calculators and other electronic or communication devices are not permitted in this examination.

STATIONERY REQUIREMENTS	**SPECIAL REQUIREMENTS**
6 gold cover sheets and treasury tags	None
Green master cover sheet	
Script paper	
Rough paper	

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1A Consider the coordinate system (ξ, θ, ϕ) defined in terms of Cartesian coordinates (x, y, z) by

$$\begin{aligned} x &= \sinh\xi \sin\theta \cos\phi\,, \\ y &= \sinh\xi \sin\theta \sin\phi\,, \\ z &= g(\xi) \cos\theta\,, \end{aligned}$$

where $g(\xi)$ is a function to be determined, $0 \leqslant \xi < \infty$, $0 \leqslant \theta \leqslant \pi$ and $-\pi < \phi \leqslant \pi$.

(a) Determine the Cartesian components of the vectors $\boldsymbol{h}_\xi$, $\boldsymbol{h}_\theta$ and $\boldsymbol{h}_\phi$ such that the Cartesian differential $\boldsymbol{dx}$ is given by

$$\boldsymbol{dx} = \boldsymbol{h}_\xi d\xi + \boldsymbol{h}_\theta d\theta + \boldsymbol{h}_\phi d\phi\,.$$

[5]

(b) Find a suitable form for $g(\xi)$ such that $g(0) = 1$ and the coordinate system (ξ, θ, ϕ) is orthogonal. Sketch the surface $\xi = 1$. [5]

(c) The volume V is the region in which $x, y, z \geqslant 0$ and $\xi \leqslant 1$. Determine the integral

$$I = \int_S \boldsymbol{F} \cdot \boldsymbol{dS}\,,$$

where S is the surface of V and $\boldsymbol{F} = \left(\ln(x^2 + y^2), \ln(x^2 + y^2), \dfrac{z}{\sqrt{x^2 + y^2}}\right)$. [10]

2A

Consider the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = (1 + \sin y)u\,, \qquad (*)$$

where the function $u(x, y)$ is real-valued and x, y represent two-dimensional space.

(a) By separation of variables, find the eigenfunctions of $(*)$. Comment on the different forms the eigenfunctions can take. [10]

(b) Find the solution on the strip $-1 \leqslant x \leqslant 1$, $y \geqslant 0$ that satisfies the boundary conditions $u(x = \pm 1, y) = 0$, $u(x, y = 0) = 1 - x^2$ and $u(x, y) \to 0$ as $y \to \infty$. [10]

3C

(a) Consider the differential equation

$$\frac{d^2x}{dt^2} = 4x + f(t)\,, \qquad (*)$$

subject to the boundary conditions that $x \to 0$ in the limits when $t \to \pm\infty$.

Derive the Green's function associated with this differential equation, and show that it takes the form

$$G(t,t') = Ce^{A|t-t'|}\,,$$

where C and A are constants to be determined. Using this Green's function, write down the general solution to the original differential equation $(*)$ with arbitrary f. [8]

(b) Using the Green's function from part (a), evaluate $x(t)$ for

$$f(\tau) = \begin{cases} \tau, & \text{for } 0 < \tau < 1, \\ 0, & \text{otherwise,} \end{cases}$$

in each of the following regions:

(i) $t < 0$; [4]

(ii) $t > 1$; [4]

(iii) $0 < t < 1$. [4]

4B

The Fourier transform $\widetilde{f}(k)$ of a function $f(x)$ is given by

$$\widetilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx \, .$$

(a) Show that if $g(x) = e^{-x^2/2}$ then $\widetilde{g}(k) = c g(k)$ for some constant c. Show also that $g^{(m)}(x) = r_m(x) g(x)$ for some polynomial r_m of degree m. [Here, and below, $y^{(m)}$ represents the m^{th} derivative of y.] [6]

(b) Prove that $\widetilde{(f^{(m)})} = (ik)^m \widetilde{f}$ and $\widetilde{(x^m f)} = i^m (\widetilde{f})^{(m)}$. Show that if p_n is a polynomial of degree n, then the Fourier transform of $p_n(x) e^{-x^2/2}$ is $q_n(k) e^{-k^2/2}$, where q_n is also a polynomial of degree n. [8]

(c) Show that if f satisfies the equation

$$f''(x) - x^2 f(x) = \mu f(x) \, , \quad x \in \mathbb{R} \, , \qquad (*)$$

where μ is a constant, then its Fourier transform $\widetilde{f}$ satisfies the equivalent equation

$$(\widetilde{f})''(k) - k^2 \widetilde{f}(k) = \mu \widetilde{f}(k) \, , \quad k \in \mathbb{R} \, .$$

[2]

(d) For each $n \geqslant 0$, there is a polynomial p_n of degree n, unique up to multiplication by a constant, such that the function

$$g_n(x) = p_n(x) e^{-x^2/2}$$

is a solution to $(*)$ for some $\mu = \mu_n$. Using this fact, along with your results from (b) and (c), deduce that $\widetilde{g}_n(k) = c_n g_n(k)$ for some constants c_n.

[4]

5B

Let M be a real $n \times n$ anti-symmetric matrix, where $n \geqslant 3$.

(a) Show that M^2 is a real symmetric non-positive matrix, i.e.

$$\boldsymbol{x}^T M^2 \boldsymbol{x} \leqslant 0, \qquad \forall \boldsymbol{x} \in \mathbb{R}^n .$$

Prove that if n is odd, then M^2 has at least one vanishing eigenvalue, and that for any n, if $M^2\boldsymbol{w} = 0$ where $\boldsymbol{w} \neq 0$, then also $M\boldsymbol{w} = 0$. [4]

(b) Let $\boldsymbol{u}_1$ be a unit eigenvector of M^2 with eigenvalue $\lambda_1 = -\mu_1^2 < 0$. Show that $\boldsymbol{v}_1 = \frac{1}{\mu_1} M\boldsymbol{u}_1$ is also a unit eigenvector of M^2, with the same eigenvalue and orthogonal to $\boldsymbol{u}_1$. [4]

(c) Explain briefly why there is always a third eigenvector $\boldsymbol{u}_2$ of M^2 (with an eigenvalue λ_2) that is orthogonal to $\boldsymbol{u}_1$ and $\boldsymbol{v}_1$. Prove that if $\lambda_2 = -\mu_2^2 < 0$, then $\boldsymbol{v}_2 = \frac{1}{\mu_2} M\boldsymbol{u}_2$ is also a unit eigenvector for λ_2, and that $\boldsymbol{v}_2$ is orthogonal to $\boldsymbol{u}_1, \boldsymbol{v}_1, \boldsymbol{u}_2$. Hence, determine the maximal number of distinct eigenvalues of an $n \times n$ matrix M^2, where M is anti-symmetric. [6]

(d) Express $M\boldsymbol{u}_1$ and $M\boldsymbol{v}_1$ in terms of $\boldsymbol{u}_1$ and $\boldsymbol{v}_1$, hence show that a 3×3 anti-symmetric real matrix M can we written in the form $M = WBW^T$, where

$$B = \begin{bmatrix} 0 & -\mu_1 & 0 \\ \mu_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and W is an orthogonal 3×3 matrix. [6]

6B

(a) Explain how to diagonalize a real symmetric matrix A. [3]

(b) Describe the quadratic surface Σ in $\mathbb{R}^3$ defined by

$$5x_1^2 - 8x_1x_2 + 5x_2^2 + 9x_3^2 - 9 = 0\,,$$

specifying the principal axes and, if appropriate, the semi-axis lengths. [9]

(c) Show that Σ intersects the spherical surface S defined by

$$x_1^2 + x_2^2 + x_3^2 = 4$$

in a pair of circles. For each circle, find its radius and the position of its centre. Show that the line connecting the two centres is orthogonal to the planes of each circle. [8]

7B

(a) State and prove the Cauchy-Riemann equations for an analytic function

$$f(z) = u(x, y) + iv(x, y)\,,$$

where $z = x + iy$ and x, y, $u(x, y)$, $v(x, y) \in \mathbb{R}$. [4]

(b) Determine the analytic function $f(z)$ whose real part is

$$u(x, y) = x \cos x \cosh y + y \sin x \sinh y\,.$$

[You should express your answer in terms of z.] [6]

(c) Find the power series expansion of the function

$$f(z) = (z + z^{-1})e^{1/z}$$

about $z = 0$, and determine the type of singularity and the residue of f at $z = 0$. [5]

(d) Find the radii of convergence about $z = 1$ for:

(i) The Taylor series of

$$f(z) = \frac{1}{z^2 + 2}\,;$$

[2]

(ii) The power series

$$\sum_{n=0}^{\infty} \left(\frac{n}{n+\alpha}\right)^{n^2} (z-1)^n\,, \quad \alpha > 0\,.$$

[3]

8C

Consider the second-order differential equation:

$$(z+z^3)^2\,\frac{d^2y}{dz^2}+3z\frac{dy}{dz}+y=0\,.$$

(a) For which values of $z \in \mathbb{C}$ is this differential equation singular? Determine whether these singularities are regular or irregular. [6]

(b) Determine how many (linearly independent) power series solutions there are when expanding the differential equation about $z = 0$. Write down the recurrence relation(s) for their coefficients. [10]

(c) For each power series solution identified in part (b), calculate the first four nonzero coefficients up to an overall multiplicative constant. [4]

9C

(a) Consider a light ray propagating in the (x, y) plane in Cartesian coordinates (x, y, z).

(i) Explain what is meant by *Fermat's Principle*. [2]

(ii) Suppose a mirror is located on the $y = 0$ plane. The light ray hits the mirror and reflects from it. Using Fermat's principle, show that the angle of incidence θ_i equals the angle of reflection θ_r. Show that this solution is unique. [*Hint: The reflected ray remains in the (x, y) plane so you can treat the problem as two-dimensional.*] [8]

(b) A function $f(x)$ is constrained such that $f(-1) = f(1) = 0$. Determine $f(x)$ such that the functional

$$I=\int_{-1}^{1}\left(\frac{df}{dx}+f+x\right)\left(\frac{df}{dx}+2f-2x\right)dx$$

is stationary. [10]

10B

Consider the Sturm-Liouville eigenvalue equation

$$-\left[p(x)y'(x)\right]' + q(x)y(x) = \lambda w(x)\,, \qquad x \in [a,b]\,, \qquad (*)$$

where $p(x) > 0$, $q(x) > 0$, $w(x) > 0$ for $x \in [a,b]$, and primes denote differentiation with respect to x.

(a) Show that, for particular boundary conditions (which you should specify), finding the eigenvalues λ in $(*)$ is equivalent to finding the stationary values of the functional

$$\Lambda(y) = \frac{F(y)}{G(y)}\,,$$

where

$$F(y) = \int_a^b (py'^2 + qy^2)\,dx\,, \qquad G(y) = \int_a^b wy^2\,dx\,.$$

[6]

(b) A function $\hat{y}$ can be expanded as

$$\hat{y}(x) = \sum_{n=0}^{\infty} c_n y_n(x)\,,$$

where c_n are constants and y_n are eigenfunctions of $(*)$, orthonormal with respect to the inner product $\langle f, g\rangle = \int_a^b f^* g\, w\, dx$, and with ordered eigenvalues λ_n, i.e. $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$. Using this fact, explain the Rayleigh-Ritz method for estimating the lowest eigenvalue λ_0. [6]

(c) The wavefunction $y(x)$ for a quantum harmonic oscillator satisfies

$$\left[-\frac{d^2}{dx^2} + x^2\right] y = \lambda y, \quad -\infty < x < \infty\,, \qquad y(x) \to 0 \quad (x \to \pm\infty)\,.$$

Use the trial function

$$\hat{y}(x) = \begin{cases} \cos(\gamma x), & |x| \leqslant \dfrac{\pi}{2\gamma}\,, \\ 0, & \text{otherwise,} \end{cases}$$

to estimate the lowest eigenvalue λ_0. [8]

END OF PAPER