UNIVERSITY OF CAMBRIDGE

CST0/MATH/2

NST0

NATURAL SCIENCES TRIPOS Part IA

Wednesday, 14 June, 2023 9:00 am to 12:00 pm

MATHEMATICS (2)

Read these instructions carefully before you begin:

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

You may submit answers to **all** of section A, and to no more than **five** questions from section B.

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on one side of the paper only and begin each answer on a separate sheet. (For this purpose, your section A attempts should be considered as one single answer.)

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Tie up your section A answer in a **single** bundle, with a completed gold cover sheet.

Tie up **each** section B answer in a **separate** bundle, marked with the question number. **Do not join the bundles together**. For each bundle, a gold cover sheet **must** be completed and attached to the bundle.

A **separate** green master cover sheet listing all the questions attempted **must** also be completed. (Your section A answer may be recorded just as A: there is no need to list each individual short question.)

Every cover sheet must bear your blind grade number and desk number, and should not include your name.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
6 gold cover sheets and treasury tags	No calculators may be used.
Green master cover sheet	No electronic devices may be used.
Script paper	

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1

Find X and Y in terms of one of the sets $\emptyset$, Ω, A, $\overline{A}$, if

(a) $A \cap \Omega = X\,,$ [1]

(b) $(A \cap \overline{A}) \cup (A \cup \overline{A}) \cup \overline{A} = Y\,,$ [1]

where A is a subset in the sample space Ω.

2

The Cartesian coordinates of point A are $(x, y, z) = (-3, -4, -1)$. For this point A, find

(a) cylindrical polar coordinates, (r, θ, z), [1]

(b) spherical polar coordinates, (r, θ, ϕ), where the angle θ is measured from the positive z-axis. [1]

3

Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be vectors in three-dimensional space.

(a) What are the expressions for α, β and γ in $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$? [1]

(b) Give the geometrical interpretation of $|\mathbf{a} \times \mathbf{b}|$. [1]

4

Find the vector area $\mathbf{S}_{ABCA}$ of the triangle $\triangle ABC$ with vertices at $A(0, 0, 0)$, $B(1, 1, 0)$ and $C(1, 1, 1)$, travelling round the perimeter in the direction $A \to B \to C \to A$. [2]

5

Let

$$y(x) = \int_{-\infty}^{x} e^{-u^2}\, du\,.$$

Sketch the graph of $y(x)$ and label any points of intersection with the axes. [2]

6

Consider using the Newton–Raphson method to solve the equation $f(x) = 0$ for real variable x.

(a) Let x_0 be an initial guess for the solution. Write down the formula for x_1, the next approximation to the solution, in terms of x_0, $f(x_0)$ and $f'(x_0)$. [1]

(b) Find the value of x_1 for the particular case of $f(x) = x - 2 + \ln x$ and $x_0 = 1$. [1]

7

For $\Phi(x, y, z) = \frac{1}{2}\left[(x-1)^2 + (y-1)^2 + (z-1)^2\right]$, find at the origin $(0, 0, 0)$:

(a) $\boldsymbol{\nabla}\Phi$, [1]

(b) the value of the directional derivative of Φ in the direction of $(1, 1, 0)$. [1]

8

(a) Evaluate $\iint\limits_D xy\, dx\, dy$, where the area of integration D is the interior of the unit circle centred at the origin. [1]

(b) Evaluate the surface integral $\oint\limits_S (\nabla \times \nabla\Phi) \cdot d\mathbf{S}$, where the surface of integration S is the surface of the unit sphere centred at the origin and $\Phi = x^2 + y^2 + z^2$. [1]

9

Rewrite

(a) $a_i \delta_{ik} b_m a_m c_k$ [1]

(b) $\delta_{jj} b_k b_k + \delta_{ii}$ [1]

in vector form, where all the indices can take values 1, 2 and 3, so that x_1, x_2 and x_3 are components of a vector $\mathbf{x}$, and summation over repeated indices is assumed, so that, for example, $A_{ij}x_j \equiv \sum_{j=1}^{j=3} A_{ij}x_j$.

10

Consider the following set of simultaneous equations:

$$\begin{cases} x + y + z &= 0 \\ 2x + 2y + 2z &= 0 \\ -x - y - z &= 0 \end{cases}$$

(a) Find all solutions of this set of simultaneous equations. [1]

(b) Give the geometrical interpretation of the solutions. [1]

SECTION B

11T

(a) Let $\mathbf{a} = \hat{\mathbf{i}}, \mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\mathbf{c} = \hat{\mathbf{k}}$, where $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are the Cartesian unit vector basis of the three-dimensional space. Show that

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) .$$

[5]

(b) Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three-dimensional vectors. Show that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ do not lie in the same plane if

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} \neq 0 .$$

[3]

(c) Determine and justify your answer whether:

(i) the four points with position vectors $\mathbf{P}_1 = (0, 0, 2)$, $\mathbf{P}_2 = (0, 1, 3)$, $\mathbf{P}_3 = (1, 2, 3)$ and $\mathbf{P}_4 = (2, 3, 4)$ lie in the same plane, [6]

(ii) the four points with position vectors $\mathbf{Q}_1 = (-2, 1, 1)$, $\mathbf{Q}_2 = (-1, 2, 2)$, $\mathbf{Q}_3 = (-3, 3, 2)$ and $\mathbf{Q}_4 = (-2, 4, 3)$ lie in the same plane. [6]

12R

Consider the function

$$h(x, y) = ae^x + be^y - cx - dy ,$$

where a, b, c and d are non-zero real constants such that $ac > 0$ and $bd > 0$.

(a) Find the stationary points and stationary values of the function. [4]

(b) Determine the nature of the stationary points. [4]

(c) Sketch the contours of the function in the range $|x| \leqslant 2$ and $|y| \leqslant 2$ for the cases

(i) $a = b = c = d = 1$,

(ii) $a = c = 1 , \quad b = d = -1$.

[6]

(d) For each case considered in part (c), find $\boldsymbol{\nabla} h$ and add arrows to your sketch showing directions of $\boldsymbol{\nabla} h$ near the stationary points. [6]

13Z

(a) Consider the vector fields

$$\mathbf{F} = \begin{pmatrix} 4x - 2y - 2z \\ -2x + 2y + az \\ -2x + ay + 2z \end{pmatrix}, \qquad \mathbf{G} = \begin{pmatrix} y\cos(xy) - by + (b+c)z \\ x\cos(xy) + bx - bz \\ (c-b)x + by \end{pmatrix},$$

where a, b and c are real constants. Explicitly find the line integrals of each of these fields in turn along the following two paths:

(i) The straight lines connecting the points $(0,0,0)$ to $(1,1,0)$, followed by $(1,1,0)$ to $(1,1,1)$, [9]

(ii) and then separately the curve from $(0,0,0)$ to $(1,1,1)$ on which the position vector is parameterized by

$$\mathbf{x}(t) = (t, t, t^2)$$

for $0 \leqslant t \leqslant 1$. [6]

(b) For what values (if any) of the constants a, b and c are the vector fields $\mathbf{F}$ and $\mathbf{G}$ given in part (a) conservative?

[5]

14X

Marks are allocated for correct numerical solutions together with explanation of how those solutions arise.

(a) Letters $ABCDEF$ are written in random order but without repetition into places 1, 2, 3, 4, 5 and 6.

(i) How many distinct arrangements exist?

(ii) How many distinct arrangements show F in 6th place?

(iii) How many distinct arrangements show either E or F in 6th place?

(iv) How many distinct arrangements show E in 5th place and F in 6th place?

(v) How many distinct arrangements show E in 5th place or F in 6th place including cases where both requirements are satisfied?

(vi) How many distinct arrangements show E in 5th place or F in 6th place excluding cases where both requirements are satisfied? [10]

(b) The letters $ABCDEF$ are partitioned (separated) into two bins as follows. Four letters are chosen at random but without repetition and placed in the first bin in which their order does not matter. The remaining two letters are placed in the second bin and again their order within the bin does not matter. This partition is denoted as $\{4, 2\}$. Therefore $ABCD$-EF is taken to be identical to $BADC$-FE but both are distinct from $ABCE$-DF.

(i) How many distinct $\{4, 2\}$ partitions show A and B in the second bin?

(ii) In the sample space of part a(i), how many times does the partition $ABCD$-EF appear? Assume that the first bin receives the letters from places $1, 2, 3$ & 4.

(iii) How many distinct $\{4, 2\}$ partitions exist?

(iv) Calculate the product of your answers to parts b(ii) and b(iii) and explain the value you obtain.

(v) Enumerate the seven distinct partitions (for example, $\{0, 6\}$ and $\{1, 5\}$) and for each perform calculations similar to those of parts b(ii) to b(iv). [10]

15Y

(a) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + y = \cos(kx)\,,$$

where k is a positive integer. [10]

(b) (i) Rewrite the set of simultaneous differential equations for functions $u(t)$ and $v(t)$,

$$\begin{cases} \frac{du}{dt} & = au + bv\,, \\ \frac{dv}{dt} & = cu + dv\,, \end{cases}$$

in the form

$$\frac{d^2u}{dt^2} - \alpha\frac{du}{dt} + \beta u = 0\,,$$

where you need to find α and β in terms of real parameters a, b, c and d. [4]

(ii) Find $u(t)$ and $v(t)$ for the particular choice of $a = 3$, $b = -1$, $c = 1$, $d = 1$ and subject to the conditions $u(0) = 0$ and $v(0) = 1$. [6]

16V

(a) Without using the divergence theorem, calculate the flux of the vector field

$$\mathbf{F} = |x|\;\hat{\mathbf{i}} + 2xy\,\hat{\mathbf{j}} + e^{-z}\,\hat{\mathbf{k}}$$

through the surface of the axes-aligned unit cube, two of whose vertices are at $(1, 0, 0)$ and $(2, 1, 1)$. [6]

(b) (i) For the surface S with $0 \leqslant x \leqslant 2\pi$, $0 \leqslant z \leqslant 1$ and $y = (1 + z)$, express the surface area element dS in terms of dx and dz by projecting the area onto the (x, z)-plane. [3]

(ii) Hence evaluate

$$\iint_S (x + yz)\,dS\,.$$

[4]

(c) By using surface integration, calculate the flux of the vector field $\mathbf{F}$ through the surface of the unit sphere centred at the origin in each of the following cases:

(i) $\mathbf{F} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$, [4]

(ii) $\mathbf{F} = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$. [3]

17T

(a) Let $\mathbf{B}$ be a 3×3 real matrix satisfying

$$\mathbf{x}^{\mathrm{T}}\mathbf{B}\mathbf{x} \geqslant |\mathbf{x}|^2$$

for every three-dimensional vector $\mathbf{x}$. Show that $\mathbf{B}$ is a non-singular matrix. [4]

(b) Find a 3×3 orthogonal matrix $\mathbf{Q}$ in which the first column is a multiple of the vector $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and the second column is a multiple of the vector $\mathbf{b} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$. [4]

(c) Let

$$\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{P} = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}.$$

(i) Compute $\mathbf{P}^{-1}$. [4]

(ii) Show that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix. [4]

(iii) Show that $\mathbf{A}^n = \mathbf{P}\,\mathbf{D}^n\,\mathbf{P}^{-1}$ for every positive integer n. [2]

(iv) Compute $\mathbf{A}^6$. [2]

18R

(a) Show that for integers $n > 1$

$$\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx + C_1\,,$$

$$\int \cos^n x\,dx = +\frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x\,dx + C_2\,,$$

where C_1 and C_2 are arbitrary constants. Hence find $\int \sin^6 x\,dx$ and $\int \cos^6 x\,dx$. [12]

(b) Using the above, show that

$$\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx\,,$$

$$\int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x\,dx\,.$$

Hence evaluate $\int_0^{\pi/2} \sin^2 x\,dx$, $\int_0^{\pi/2} \sin^4 x\,dx$, $\int_0^{\pi/2} \cos^2 x\,dx$ and $\int_0^{\pi/2} \cos^4 x\,dx$. [8]

19Z*

(a) By considering the integral defined on $x \in [\alpha, \beta]$

$$\int_{\alpha}^{\beta} (f(x) + \lambda g(x))^2 \, dx$$

for continuous, real-valued functions $f(x)$ and $g(x)$ together with an arbitrary real constant λ, deduce the Schwarz inequality. Under what circumstances is this an equality?

[8]

(b) Consider a right cylindrical chamber, C, with radius a and height h.

Each concentric cylindrical shell within the chamber (i.e. the volume from r to $r + dr$, where r is the distance from the symmetry axis in cylindrical polar coordinates) moves independently around the symmetry axis at an angular speed of $\omega(r)$ such that the linear speed at any point is given by $v(r) = r\omega(r)$.

(i) A function $k = k(r)$ is defined by $k = \frac{1}{2}v^2$. Find $K = \int_C k(r)\, dV$, expressing your answer in the form of a single integral.

(ii) Another function $l = l(r)$ is defined by $l = rv$. Find $L = \int_C l(r)\, dV$, again expressing your answer in the form of a single integral.

Hence show that the Schwarz inequality implies that

$$K \geqslant \frac{L^2}{Va^2}\,, \qquad (\dagger)$$

where V is the volume of the cylindrical chamber. [9]

(iii) Find the functional form of $\omega(r)$ that renders (†) an equality. [3]

20V*

By using the method of separation of variables, find a non-constant solution for each of the following first-order partial differential equations for real-valued functions $u(x, y)$:

(a) $\dfrac{\partial u}{\partial x} + (6 - 2y)\dfrac{\partial u}{\partial y} = 0$ [6]

(b) $e^x \dfrac{\partial u}{\partial x} - xy\dfrac{\partial u}{\partial y} = 0$ [6]

(c) $\dfrac{1}{\sin x}\dfrac{\partial u}{\partial x} - \tan y \dfrac{\partial u}{\partial y} = 0$ with conditions $u(\pi/6, \pi/6) = \dfrac{1}{4e^{\sqrt{3}}}$ and $u(\pi/3, \pi/3) = \dfrac{3}{4e}$ [8]

END OF PAPER