UNIVERSITY OF CAMBRIDGE

CST0/MATH/1

NST0

NATURAL SCIENCES TRIPOS Part IA

Monday, 12 June, 2023 9:00 am to 12:00 pm

MATHEMATICS (1)

Read these instructions carefully before you begin:

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

You may submit answers to **all** of section A, and to no more than **five** questions from section B.

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on one side of the paper only and begin each answer on a separate sheet. (For this purpose, your section A attempts should be considered as one single answer.)

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Tie up your section A answer in a **single** bundle, with a completed gold cover sheet.

Tie up **each** section B answer in a **separate** bundle, marked with the question number. **Do not join the bundles together**. For each bundle, a gold cover sheet **must** be completed and attached to the bundle.

A **separate** green master cover sheet listing all the questions attempted **must** also be completed. (Your section A answer may be recorded just as A: there is no need to list each individual short question.)

Every cover sheet must bear your blind grade number and desk number, and should not include your name.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
6 gold cover sheets and treasury tags	No calculators may be used.
Green master cover sheet	No electronic devices may be used.
Script paper	

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1

Determine the numerical values of the coefficients of x^4 and x^5 in the expansion of

$$(1+x)^8 .$$

[2]

2

Given that $x = \dfrac{3}{2}$ is a root of the cubic polynomial

$$2x^3 - 5x^2 + x + 3 ,$$

find the other two roots.

[2]

3

Find all solutions for x and y of the simultaneous equations

$$\begin{cases} x + y &= 1 , \\ x - x^2 + 2y^2 &= 2 . \end{cases}$$

[2]

4

Find the radius and the coordinates of the centre of the circle

$$x^2 + y^2 + 2x + 4y = 4 .$$

[2]

5

Find the values of the parameters A and ϕ, with $A > 0$ and $0 \leqslant \phi < 2\pi$, such that

$$\cos t + \sqrt{3}\,\sin t = A \sin(t + \phi)$$

for all real values of t.

[2]

6

Find integers p and q such that

$$\sum_{n=1}^{10} 2^{-(n/2)} = \frac{p + q\sqrt{2}}{32}\,.$$

[2]

7

Find the stationary values of the function

$$f(x) = \exp\left(\frac{x}{1+x^2}\right)$$

and the values of x at which they occur.

[2]

8

Given that

$$x + y + e^x + e^y = c\,,$$

where c is a constant, find $\dfrac{dy}{dx}$ in terms of x and y.

[2]

9

Show that

$$\int_2^3 \frac{2x+1}{x(x+1)}\,dx = \ln n$$

for some integer n that you should determine.

[2]

10

Calculate the finite area enclosed between the graphs $y = 2x$ and $y = 4x\,e^{-x}$.

[2]

SECTION B

11Y

(a) Find the real and imaginary parts of

(i)

$$(3-2i)(4+3i)\,,$$

[2]

(ii)

$$\frac{3-2i}{4+3i}\,.$$

[2]

(b) Find and sketch the set of points in the Argand diagram that satisfy the following equations with $z = x + iy$:

(i)

$$\mathrm{Re}(z^2) = \mathrm{Im}(z^2)\,,$$

[4]

(ii)

$$\frac{\mathrm{Im}(z^2)}{z^2} = -i\,.$$

[4]

(c) Solve, for real x, the equation

$$\cosh x = \sinh x + 2\,\mathrm{sech}\,x\,.$$

[4]

(d) Find all distinct solutions for z of the equation

$$\cosh z = i\,.$$

[4]

12V

(a) (i) A solid metal object has height H and a circular base of radius R. The radius of the horizontal cross-section of the object at height z above the base reduces with z $(0 \leqslant z \leqslant H)$ according to the formula $R\,e^{-z/H}$. Additionally, the cross-section is a sector of the circle: the full circle (2π radians) at the base but reducing linearly with height to π radians at height $z = H$. The figure shows the horizontal cross-section of the object at the base, at the mid-height, and at the top. Find the volume of the object. [7]

(ii) Material is removed from the middle of the object by cutting a hole of the same shape and height, but only 80% of the radius, from inside the object. The removed metal is reformed as a solid sphere of the same volume. Find the radius, a, of the sphere in terms of R and H. [3]

(b) A second object has a triangular base in the (x, y)-plane. One edge of the triangular base runs along the x-axis from $-L$ to $+L$ and the third vertex is at $(x, y, z) = (0, D, 0)$. The object is symmetric about the (y, z)-plane and attains its maximum height (measured along the z-axis) at $(x, y, z) = (0, 0, H)$. The cross-section remains triangular but the width of the object (parallel to the x-axis) varies according to $w(z) = 2L\left(1 - \frac{z}{2H}\right)^{1/2}$. The depth of the object (parallel to the y-axis) decreases linearly with height from D at $z = 0$ to 0 at $z = H$.

(i) Sketch the projections of the object on to the (x, y)-plane, the (x, z)-plane and the (y, z)-plane. [4]

(ii) Calculate the volume of the object. [6]

13Y

(a) Find explicitly the general solution $y(x)$ of the differential equation

$$x^2 \frac{dy}{dx} + xy^2 = 4y^2 \,.$$

[4]

(b) Find explicitly the solution $y(x)$ of the differential equation

$$\frac{dy}{dx} - y \tan x = 1$$

subject to the condition

$$y(\pi/4) = 3 \,.$$

[4]

(c) For the differential equation

$$\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right) \,,$$

find explicitly:

(i) the general real-valued solution for $y(x)$, [7]

(ii) the particular solution satisfying $y(1) = \pi/3$, [2]

(iii) the particular solution satisfying $y(1) = 2\pi/3$. [3]

14R

(a) Consider the differential form $P\,dx + Q\,dy$, where

$$P(x,y) = \frac{1+\sqrt{1+y^2}}{a\sqrt{x}}\,, \qquad Q(x,y) = \frac{y\sqrt{x}}{\sqrt{1+y^2}}\,,$$

$x > 0$ and a is a non-zero real parameter. Find all values of the parameter a for which this differential form can be written as an exact differential, i.e. $P\,dx + Q\,dy = dv$, of some function $v(x,y)$. Hence find the most general function $v(x,y)$ for the corresponding value(s) of a. [6]

(b) The differential of the volume V of a geometrical figure is given by

$$dV = (2\pi r h)\,dr + (\pi r^2)\,dh\,,$$

where r and h are non-negative parameters and the volume vanishes when these parameters are zero. Find an expression for the fractional change in volume dV/V for fractional changes in the parameters dr/r and dh/h. Find dV/V if r increases by 1% and h increases by 2%. [4]

(c) Consider the function $u(x,y) = x\,\phi(y/x)$, where ϕ is a differentiable function of its argument and $x \neq 0$. Show that u satisfies

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u\,.$$

[4]

(d) Consider the function $u(x,y)$ which is twice-differentiable in its arguments and in arguments ξ and η obtained by the following linear transformation of x and y:

$$\xi = ax - by\,, \quad \eta = bx + ay\,,$$

where a and b are real constants and $a^2 + b^2 = 1$. Show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2}\,.$$

[6]

15S

Find, by any method, the first three non-zero terms in the Taylor series expansions about $x = 0$ of the following functions. [You may quote standard power series without proof.]

(a) $x\,\sinh(x^2)$ [4]

(b) $\ln\big(1 + \ln(1 + x)\big)$ [8]

(c) $\sin^6 x$ [8]

16X

Measurements of a continuous random variable X show that the values obtained always lie in the range $0 \leqslant x \leqslant 1$. The probability $F(x)$ of finding values less than or equal to x is given, for $0 \leqslant x \leqslant 1$, by

$$F(x) = A\left(\frac{x^3}{3} - \frac{x^4}{4}\right) .$$

(a) Calculate the value of A. [2]

(b) Sketch $F(x)$ in the range $-1 \leqslant x \leqslant 2$. [3]

(c) Find the probability density function $f(x)$ for X. [2]

(d) Sketch $f(x)$ in the range $-1 \leqslant x \leqslant 2$. [3]

(e) The mode is the value of x at which the probability density has its maximum value. Calculate the mode and mark it on your sketch. [2]

(f) The median is the value of x at which $F(x) = 1/2$. Is the median greater than or less than the mode? [2]

(g) Calculate the mean, $E[X]$, and the standard deviation, σ. [3]

(h) What is the probability that X takes a value between $E[X] - \sigma$ and $E[X] + \sigma$? Leave your answer in terms of rational fractions but do not try to evaluate them. [3]

17Z

(a) State what is meant for two real-valued functions $f(x)$ and $g(x)$ to be *orthogonal* on an interval $-L \leqslant x \leqslant L$. [2]

(b) The basis set of trigonometric functions

$$\cos\frac{n\pi x}{L}\,, \qquad \sin\frac{n\pi x}{L}\,,$$

for non-negative integer n, are orthogonal over the interval $-L < x \leqslant L$. It is given that the function $h(x)$ can be written as a linear combination of such functions, i.e.

$$h(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right].$$

Give expressions for the appropriate coefficients of the expansion of h in terms of this basis as integrals involving the function h. [2]

(c) A function $h(x)$ is defined as

$$h(x) = x^2 \quad \text{for } -1 < x \leqslant 1\,.$$

Find the expansion of h in terms of the trigonometric functions defined in part (b), with $L = 1$. [6]

(d) Hence find the expansion, in terms of the trigonometric functions defined in part (b), of the function $g(x)$, where

$$g(x) = x \quad \text{for } -1 < x \leqslant 1\,.$$

[2]

(e) Using your results from parts (c) and (d) above, deduce integers A and B such that

$$\pi = \sum_{n=0}^{\infty}\frac{A(-1)^n}{2n+1} \qquad \text{and} \qquad \pi^4 = \sum_{n=1}^{\infty}\frac{B}{n^4}\,.$$

[8]

[You might find it useful to note that for a function defined as in part (b) we have

$$\frac{1}{L}\int_{-L}^{L}\big(h(x)\big)^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)\,.]$$

18T

Let

$$\mathbf{A} = \begin{pmatrix} 2\alpha+\gamma & \gamma & 2\beta \\ 2\beta & \alpha & \beta+\gamma \\ \beta+\gamma & \alpha+3\beta & \alpha+\gamma \end{pmatrix}$$

be a matrix depending on the real parameters α, β, γ . Let $\mathbf{c}$ and $\mathbf{b}$ be the vectors

$$\mathbf{c} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix} .$$

(a) Find the values of the parameters α, β, γ such that

$$\mathbf{Ac} = \mathbf{b} \,. \qquad (\dagger)$$

If there is more than one solution for α, β, γ express β and γ in terms of α. [8]

(b) Express the matrix $\mathbf{A}$ that satisfies (†) in terms of α and denote the answer by $\mathbf{A}_\alpha$. [2]

(c) Compute $\det(\mathbf{A}_\alpha)$. [5]

(d) Show that the matrix $\mathbf{A}_\alpha$ is non-singular for every real α. [2]

(e) Show that for every real α the matrix $\mathbf{A}_\alpha$ has at least one real eigenvalue which is positive. [3]

19W*

(a) State without proof the ratio test for series convergence. [2]

(b) By using any test for series convergence, determine whether the following series converge or diverge:

(i) $\displaystyle\sum_{k=0}^{\infty} \frac{1}{1+k^2}\,,$ [2]

(ii) $\displaystyle\sum_{k=0}^{\infty} \frac{a^{2k+1}}{2k+1}\,,$ where $a > 0\,.$ [3]

(c) State l'Hôpital's rule for evaluation of limits of functions. [3]

(d) By using any method, find the following limits for real variable x and real parameter a:

(i) $\displaystyle\lim_{x\to 0^+} x\ln x\,,$ [2]

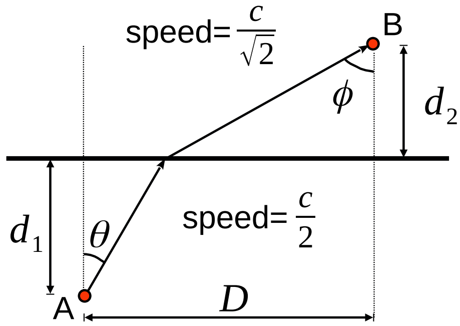
(ii) $\displaystyle\lim_{x\to a} \frac{x^x - a^a}{x-a}\,,$ where $a > 0\,,$ [2]

(iii) $\displaystyle\lim_{x\to\infty} \left(1 + a^x + \left(\frac{a^2}{2}\right)^x\right)^{\frac{1}{x}}\,,$ where $a \geqslant 0\,.$ [6]

20X*

(a) Explain how the method of Lagrange multipliers is used to find the stationary points of a function $f(x, y, z, \ldots)$ subject to a constraint $g(x, y, z, \ldots) = r$ for some constant r. Include some discussion of the theoretical or geometrical basis for the method. Explain how a second constraint could be taken into account. [6]

(b) The diagram shows the trajectory of a particle travelling between fixed points A and B. The particle travels at speed $c/\sqrt{2}$ above the fixed horizontal line and at speed $c/2$ below the line. Distances are as shown in the diagram and the problem is to find angles θ and ϕ so as to minimise the travel time.



(i) Use the method of Lagrange multipliers to find a relation between θ and ϕ. How is this relation affected if the distance parameters D, d_1 and d_2 change? [6]

(ii) Use the Lagrange constraint to find an implicit equation for $\sin\theta$ for the case $D = 2$, $d_1 = \sqrt{3}$, $d_2 = 1$. Solve this equation for $\sin\theta$ given the hint that $\operatorname{cosec}\theta$ is equal to an integer. [4]

(iii) For the values of the distance parameters defined in part (ii), calculate the values of θ, ϕ, the distance travelled, and the time taken. Make a sketch of this special case to approximate scale. [4]

END OF PAPER