UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB

Wednesday, 15 June, 2022 9:00am to 12:00pm

MATHEMATICS (2)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, ***6B****).*

Tie up ***each*** *answer in a* ***separate*** *bundle, marked with the question number.*

Do not join the bundles together.

For each bundle, a gold cover sheet ***must*** *be completed and attached to the bundle.*

A ***separate*** *green master cover sheet listing all the questions attempted* ***must*** *also be completed.*

Every cover sheet must bear your examination number and desk number.

Calculators are not permitted in this examination.

STATIONERY REQUIREMENTS
6 gold cover sheets and treasury tags
Green master cover sheet
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1C

Consider the eigenvalue problem

$$-(1-x^2)y_n'' + 3xy_n' = (n^2-1)y_n, \qquad -1 \leqslant x \leqslant 1\,, \qquad (*)$$

where $n \geqslant 1$ is an integer, and y_n and y_n' are finite at $x = \pm 1$.

(a) Rewrite this equation in Sturm-Liouville form and determine the weight function $w(x)$. [4]

(b) Using directly the Sturm-Liouville form, show that any two eigenfunctions y_n and y_m of $(*)$, with $n \neq m$, satisfy the orthogonality condition

$$\int_{-1}^{1} y_n(x) y_m(x) w(x)\, dx = 0.$$

Determine a new weight $w_1(x)$ so that the first derivatives y_n' and y_m' are orthogonal with respect to $w_1(x)$. [6]

(c) Using the substitution $x = \cos\theta$, with $\theta \in [0, \pi]$, verify that

$$y_n(x) = \frac{\sin(n \arccos x)}{\sqrt{1-x^2}}$$

is an eigenfunction of $(*)$. [6]

(d) Prove that $y_n(x)$ is an algebraic polynomial in x of degree $n-1$ for $n \geqslant 1$. *[Hint: Consider* $y_{n+1} + y_{n-1}$*.]* [4]

2C

Consider the diffusion equation

$$u_t = u_{xx}\,,$$

for $t \geqslant 0$, $x \in [0,1]$, and with the boundary conditions

$$u_x(0,t) = u(0,t), \qquad u_x(1,t) = u(1,t)\,.$$

(a) Using the method of separation of variables, show that the solution with the given boundary conditions can be written as

$$u(x,t) = c_0 e^t e^x + \sum_{n=1}^{\infty} c_n e^{-\pi^2 n^2 t} V_n(x),$$

where

$$V_n(x) = \pi n \cos(\pi n x) + \sin(\pi n x),$$

and c_0 and c_n are real constants. [15]

(b) For the initial condition $u(x,0) = f(x)$, find c_0 and c_n as integrals involving f. You may assume that the functions V_n are orthogonal to each other and to e^x on the interval $[0,1]$. [5]

3C

For a volume V in $\mathbb{R}^3$ with boundary S, we seek a solution to Poisson's equation with Dirichlet boundary conditions on S:

$$\begin{aligned} \nabla^2 u &= \rho, \quad \text{in } V\,, \\ u &= f, \quad \text{on } S\,. \end{aligned} \qquad (*)$$

(a) State the definition of a Green's function $G(\mathbf{r},\mathbf{r}')$ for this problem. Use Green's identity to show that the solution to $(*)$ can be expressed as

$$u(\mathbf{r}') = \int_V \rho(\mathbf{r}) G(\mathbf{r},\mathbf{r}') dV + \int_S f(\mathbf{r}) \frac{\partial G}{\partial n} dS\,.$$

[6]

(b) Write down the fundamental solution in $\mathbb{R}^3$, and use the method of images to determine the Green's function $G(\mathbf{r},\mathbf{r}') = G_0(\mathbf{r},\mathbf{r}')$ in the case $V = V_0$, where $V_0 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leqslant 1\}$. [6]

(c) Further, use the method of images to determine the Green's function $G(\mathbf{r},\mathbf{r}') = G_1(\mathbf{r},\mathbf{r}')$ in the case $V = V_1$, where $V_1 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leqslant 1 \text{ and } z \geqslant 0\}$. [8]

4A

(a) Consider the integral

$$I = \int_\Gamma \frac{f(z)}{z^\alpha}\, dz$$

in $z \in \mathbb{C}$, with $f(z)$ analytic everywhere, $0 < \alpha < 1$ and a branch cut along $z = re^{i\phi}$ for fixed $0 < \phi < \pi/2$ with $r \geqslant 0$. Determine the jump in the integrand across the branch cut. [4]

(b) Consider the function

$$g(z) = \frac{1}{(z^2+1)z^{1/4}}.$$

(i) Identify points in the complex plane where $g(z)$ is not analytic, stating the nature of the features identified. [3]

(ii) Evaluate the integral

$$J = \int_0^\infty g(z)\, dz$$

using contour integration around a closed contour. [13]

5A

The height of a surface is given by

$$h(x) = \begin{cases} e^{x/L} & x < 0, \\ e^{-x/L} & x \geqslant 0, \end{cases}$$

where x is the horizontal position and $L > 1$ a constant.

(a) Compute $\tilde{h}(k)$, the Fourier transform of $h(x)$. [4]

(b) The response of a new coating material is controlled by

$$\frac{d^4y}{dx^4} - 5\frac{d^2y}{dx^2} + 4y = h(x),$$

with $y \to 0$ for $x \to \pm\infty$. Determine $\tilde{y}(k)$, the Fourier transform of y, and hence determine $y(x)$. [13]

(c) Evaluate $y(0)$ in the limit $L \to 1$. [3]

6B

(a) Write down the transformation law for a general tensor of rank n. What is the definition of an *isotropic tensor*? Prove that the most general isotropic tensor of rank two is $\lambda\,\delta_{ij}$, where λ is a scalar, by considering rotations by 90° in three dimensions of a general tensor of rank two T_{ij}. [6]

(b) Consider a symmetric rank-two tensor T_{ij} and let A_i be a non-zero rank-one tensor. Find the scalars α, β, rank-one tensor B_i, and a symmetric traceless rank-two tensor C_{ij} such that

$$T_{ij} = \alpha\,\delta_{ij} + \beta\,A_i\,A_j + B_i\,A_j + B_j\,A_i + C_{ij}\,,$$

$$A_iB_i = 0\,, \quad A_iC_{ij} = 0\,.$$

[10]

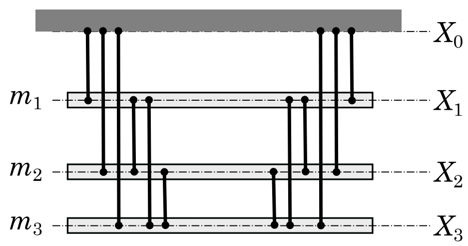
(c) Consider

$$T_{ij} = \begin{pmatrix} 10 & 20 & 30 \\ 20 & 40 & -10 \\ 30 & -10 & 50 \end{pmatrix}, \quad A_i = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}.$$

Find α, β, B_i and C_{ij} defined in part (b). [4]

7A

A creative musician wants to build a novel musical instrument comprising elastic strings suspending n weighted bars of mass m_i, $i = 1, \ldots n$ from a rigid beam. The instrument is arranged such that the beam and all the weighted bars remain parallel and are connected to each other by pairs of elastic strings, as illustrated in the figure below for $n = 3$. At time t the vertical position of the weighted bars is given by X_i ($i = 1, \ldots n$); the position of the beam is given by X_0. The strings all have the same spring constant $k/2$ and are of a type that has negligible mass and zero length when they are not extended. The elastic potential energy of a string connecting bars i and j is therefore given by $E_{ij} = \frac{1}{2}k(X_i - X_j)^2$. The bars are arranged so they cannot collide and any horizontal component of motion or force imparted by the elastic strings can be ignored.



(a) Give expressions for the gravitational potential energy and kinetic energy of each bar and hence determine the Lagrangian for the musical instrument. Determine the corresponding system of Euler-Lagrange equations. [5]

(b) The musician first explored an instrument with $n = 2$, $k = 1$ and $m_1 = 1$.

(i) Determine the eigenfrequencies ω_1, ω_2 (with $\omega_1 \leqslant \omega_2$) of the system as a function of m_2. [5]

(ii) To obtain a more pleasant sound, the musician adjusted m_2 so that $\omega_2 = 2\omega_1$. Determine the value(s) of m_2 the musician could use. [3]

(c) Next, the musician chose to explore a $n = 3$ instrument. Determine the eigenfrequencies and eigenmodes for the case $m_1 = m_2 = m_3 = 1$. [5]

(d) One of the bars in the $n = 3$ instrument becomes stuck such that $X_3 = 0$. What are the new frequencies for the instrument? [2]

8B

(a) Define the *order* of a finite group G. What is meant by a *normal subgroup* H of G? [2]

(b) Consider the 3-fold dihedral group D_3, the symmetry group of an equilateral triangle. Identify the elements of this group, explain their geometrical action on the equilateral triangle and construct the corresponding group table. Express the members of D_3 in terms of the generators. [8]

(c) Show that D_3 contains three order-2 non-normal subgroups and one order-3 normal subgroup, and show the partitioning into conjugacy classes. [4]

(d) Show that D_3 is isomorphic to the permutation group S_3. [6]

9B

Consider a finite group G and a subgroup H.

(a) Define a *left coset* of H in G. [2]

(b) Show that every coset of H contains the same number of elements. [2]

(c) Show that every order-4 group G with no element of order-4 is isomorphic to the Vierergruppe V. [6]

(d) State what is meant by a *cyclic group*. Show that if G is a non-trivial group with no proper subgroup, then G is a finite group and is cyclic of prime order. [10]

10A

Consider the complex matrix

$$Q(z_1, z_2) = \begin{bmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{bmatrix},$$

where $z_1, z_2 \in \mathbb{C}$ and z^* is the complex conjugate of z. Define $W = Q(1,0)$, $X = Q(i,0)$, $Y = Q(0,1)$, $Z = Q(0,i)$, $\hat{W} = -W$, $\hat{X} = -X$, $\hat{Y} = -Y$ and $\hat{Z} = -Z$.

(a) Determine the products WW, XX, YY, ZZ, XY, YZ and ZX. Determine also the inverses W^{-1}, X^{-1}, Y^{-1} and Z^{-1} [5]

(b) Show that $G = \{W, X, Y, Z, \hat{W}, \hat{X}, \hat{Y}, \hat{Z}\}$ is a group under matrix multiplication and determine the group table. What is the character of the group? [8]

(c) Identify three subgroups (G_1, G_2, G_3) of order 4 and provide the group table for one of these. Identify which of these groups are cyclic and how they are related to each other. [5]

(d) Find a one-dimensional representation for one of the subgroups identified in (c). [2]

END OF PAPER