UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB

Friday, 3 June, 2022 9:00am to 12:00pm

MATHEMATICS (1)

## *Before you begin read these instructions carefully:*

*You may submit answers to no more than* ***six*** *questions. All questions carry the same number of marks.*

*The approximate number of marks allocated to a part of a question is indicated in the right hand margin.*

*Write on* ***one*** *side of the paper only and begin each answer on a separate sheet.*

## *At the end of the examination:*

*Each question has a number and a letter (for example,* ***6C****).*

*Tie up* ***each*** *answer in a* ***separate*** *bundle, marked with the question number.*

***Do not join the bundles together.***

*For each bundle, a gold cover sheet* ***must*** *be completed and attached to the bundle.*

*A* ***separate*** *green master cover sheet listing all the questions attempted* ***must*** *also be completed.*

***Every cover sheet must bear your examination number and desk number.***

***Calculators are not permitted in this examination.***

***STATIONERY REQUIREMENTS***
*6 gold cover sheets and treasury tags*
*Green master cover sheet*
*Script paper*
*Rough paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1A** The interior of a paraboloidal body is defined parametrically in the Cartesian coordinate system $\boldsymbol{x} = (x, y, z)$ with

$$x = uv\cos\phi,$$
$$y = uv\sin\phi,$$
$$z = \frac{1}{2}(u^2 - v^2),$$

where $0 \leqslant u \leqslant v$, $0 \leqslant v < 1$ and $-\pi \leqslant \phi < \pi$.

(a) Sketch this body and describe its key characteristics. [4]

(b) Using $(u, v, \phi)$ as a coordinate system, determine the Cartesian components of the vectors $\boldsymbol{h}_u$, $\boldsymbol{h}_v$ and $\boldsymbol{h}_\phi$ such that the Cartesian differential $\boldsymbol{dx}$ is given by

$$\boldsymbol{dx} = \boldsymbol{h}_u du + \boldsymbol{h}_v dv + \boldsymbol{h}_\phi d\phi.$$

Determine also the corresponding scale factors. Is the coordinate system $(u, v, \phi)$ orthogonal (you must justify your answer)? [8]

(c) Determine the Jacobian for this coordinate transformation. [2]

(d) Evaluate the integral

$$I = \int_S \boldsymbol{F} \cdot \boldsymbol{dS},$$

where $S$ is the surface of the paraboloidal body, $\boldsymbol{dS}$ is an element of vector area and $\boldsymbol{F} = \boldsymbol{\nabla}\Omega$ with $\Omega = x^3 + (x^2 + y^2)z$. [6]

**2A**

A linear wave with constant frequency $\omega$ can be described in a suitably rotated and rescaled orthogonal coordinate system $(x, y)$ by the pair of equations

$$\frac{\partial b}{\partial t} + \omega \frac{\partial \psi}{\partial y} + \epsilon \left( \frac{\partial \psi}{\partial x} - \widetilde{\nabla}^2 b \right) = 0,$$

$$\frac{\partial}{\partial t} \widetilde{\nabla}^2 \psi - \omega \frac{\partial b}{\partial y} - \epsilon \frac{\partial b}{\partial x} = 0,$$

where $b = b(x, y, t)$, $\psi = \psi(x, y, t)$, $\epsilon \ll 1$ is a constant parameter and

$$\widetilde{\nabla}^2 = \epsilon^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

(a) Use the substitutions

$$b = (b_0 + \epsilon b_1 + \cdots) e^{-i\omega t},$$

$$\psi = (\psi_0 + \epsilon \psi_1 + \cdots) e^{-i\omega t},$$

to rewrite this system in the form

$$P + Q\epsilon + \mathcal{O}(\epsilon^2) = 0,$$

$$R + S\epsilon + \mathcal{O}(\epsilon^2) = 0.$$

Here, $b_i$ and $\psi_i$ (for $i = 0, 1, \ldots$) depend only on $x$ and $y$ and the functions $P$, $Q$, $R$ and $S$ may involve $b_i$, $\psi_i$ and/or derivatives of these. [5]

(b) Show that $P = R = 0$ if

$$b_0 = -i \frac{\partial \psi_0}{\partial y}.$$

[3]

(c) Set $P = Q = R = S = 0$ and eliminate $b_i$ to determine a differential equation for $\psi_0$. Show that this differential equation is satisfied when

$$\left( 2 \frac{\partial}{\partial x} + i \frac{\partial^3}{\partial y^3} \right) \psi_0 = f(x), \qquad (\ddagger)$$

where $f(x)$ is an arbitrary function of $x$. [6]

(d) Assuming that $f(x) = 0$ and using separation of variables, find the solution to $(\ddagger)$ for which $\psi_0(x = 0, y) = e^{iky}$ and $\psi_0 \to 0$ as $x \to \infty$, where $k$ is a real constant. [6]

**3B** Consider the second-order differential equation

$$\frac{d^2y(x)}{dx^2} + \frac{\alpha}{x}\frac{dy(x)}{dx} + \frac{(\alpha-1)^2}{4\,x^2}y(x) = f(x)\,, \qquad (\dagger)$$

with $\alpha$ a real constant.

(a) Find the general solution $y(x)$ to $(\dagger)$ for the case $f(x) = 0$. [6]

(b) Construct the Green's function $G(x,\xi)$ for $(\dagger)$ in the region $x \geqslant 0$ subject to the boundary conditions

$$G(0,\xi) = \left.\frac{dG(x,\xi)}{dx}\right|_{x=0} = 0\,.$$

[8]

(c) Use your Green's function to solve $(\dagger)$ for the case $f(x) = x$ in the region $x \geqslant 0$, subject to the boundary conditions

$$y(0) = \left.\frac{dy(x)}{dx}\right|_{x=0} = 0\,,$$

with $\alpha > -5$. [6]

**4C**

The Fourier transform $\widetilde{f}(k)$ of a function $f(x)$ is given by

$$\widetilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx.$$

(a) Prove that the Fourier transform of the convolution

$$h(x) = f * g := \int_{-\infty}^{\infty} f(t) g(x-t)\, dt$$

is given by $\widetilde{h}(k) = \widetilde{f}(k)\, \widetilde{g}(k)$, where $\widetilde{f}$ and $\widetilde{g}$ are the Fourier transforms of $f$ and $g$, respectively. [4]

(b) Let

$$f(x) = \begin{cases} 1, & |x| < \frac{1}{2}, \\ 0, & \text{otherwise}; \end{cases} \qquad g(x) = \begin{cases} 1 - |x|, & |x| < 1, \\ 0, & \text{otherwise}. \end{cases}$$

Show that

$$\widetilde{f}(k) = \frac{2}{k} \sin \frac{k}{2}\,, \qquad \widetilde{g}(k) = \frac{4}{k^2} \sin^2 \frac{k}{2}\,,$$

hence find the convolution of $f$ with itself. [10]

(c) State *Parseval's identity* and use the results from part (b) to evaluate

$$\int_{-\infty}^{\infty} \frac{\sin^4 x}{x^4}\, dx\,.$$

[6]

**5C**

(a) What does it mean for an $n \times n$ matrix $A$ to be diagonalizable? Show that $A$ is diagonalizable if and only if it has $n$ linearly independent eigenvectors. [4]

(b) Diagonalize the matrix

$$M = \begin{bmatrix} a & b-a & c-b \\ 0 & b & c-b \\ 0 & 0 & c \end{bmatrix},$$

where $a, b, c$ are arbitrary real numbers. [6]

(c) Let $I$ be the identity matrix. Then, for a matrix $A$ such that $A^k \to 0$ as $k \to \infty$, where 0 is the zero matrix, the following equality is true: $(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$.

Find necessary and sufficient conditions on $a, b, c \in \mathbb{R}$ that ensure $M^k \to 0$ as $k \to \infty$, hence determine for such $M$ the entries of the matrix $(I - M)^{-1}$. [7]

(d) For a set of $n$ linearly independent vectors $(\boldsymbol{x}_i)_{i=1}^n$, let $B = (B_{i,j})_{i,j=1}^n$ be the matrix with the entries

$$B_{i,j} = \boldsymbol{x}_i^T \boldsymbol{x}_j \,.$$

Prove that the quadratic form associated with the matrix $B$,

$$Q(\boldsymbol{c}) := \boldsymbol{c}^T B \boldsymbol{c} \,,$$

is positive definite, that is to say, $Q(\boldsymbol{c}) > 0$ for all non-zero $\boldsymbol{c} \in \mathbb{R}^n$. [3]

**6C**

(a) Define a *skew-Hermitian matrix* and show that its eigenvalues are purely imaginary. Define a *unitary matrix* and show that its eigenvalues have modulus 1. [4]

(b) Show that if $A$ is skew-Hermitian with distinct eigenvalues $(\lambda_i)_{i=1}^n$, then its eigenvectors are orthogonal and

$$A = UDU^\dagger,$$

where $D = \text{diag}\,(\lambda_1, \ldots, \lambda_n)$ and $U$ is unitary. [4]

(c) Let $\exp(A)$ be the matrix exponent,

$$\exp(A) := \sum_{k=0}^{\infty} \frac{A^k}{k!}\,.$$

Using (b) or otherwise, show that if $A$ is skew-Hermitian with distinct eigenvalues, then $U = \exp(A)$ is unitary. [4]

(d) Suppose that a unitary matrix $U$ can be written as $U = A + iB$, where $A$ and $B$ are real antisymmetric matrices, each with $n$ distinct eigenvalues. Show that $A$ and $B$ have the same eigenvectors and determine the eigenvalues of $A$ and $B$ in terms of eigenvalues of $U$. [8]

**7C**

(a) Write down the *Cauchy-Riemann equations* for an analytic function $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, hence show that curves of constant $u$ and curves of constant $v$ intersect at right angles. [3]

(b) Show that if $g(z)$ is analytic and $|g(z)|$ is constant, then $g$ is constant. [6]

(c) For $|z| < \infty$, find and classify the singularities of the following functions:

$$\text{(i)} \quad \cot z\,, \qquad \text{(ii)} \quad \cot\frac{1}{z}\,, \qquad \text{(iii)} \quad e^{\cot z}\,.$$

[4]

(d) Find the power series expansion (with real coefficients) about $z = 1$ of the function

$$f(z) = \frac{2z}{z^2+1}\,.$$

Determine the radius of convergence of this series. [7]

**8B**

(a) Define an *ordinary point* and a *regular singular point* of the ordinary differential equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0\,.$$

What are the implications for the existence of a series solution at such points? [6]

(b) Consider the differential equation

$$\frac{d}{dx}\left[(1-x)\,\frac{dy}{dx}\right] + \left[c^2\,(1-x) + \lambda\right]\,y = 0\,, \qquad (\dagger)$$

with $c > 0$ and $\lambda$ real.

(i) Identify the singular point(s) of (†). [2]

(ii) Show that a series solution of (†) is

$$y_1 = (1-x)^{\sigma} \sum_{n=0}^{+\infty} a_n (1-x)^n\,,$$

where you should determine $\sigma$ and find the recurrence relation for $a_n$. Why can only one solution of this form be found? [6]

(iii) Give the general solution of (†) in terms of an explicit integral involving $y_1$ and the Wronskian. [2]

(iv) Hence show that any solution of (†) that is linearly independent of $y_1$ must behave like a logarithm of $1-x$ near $x = 1$. [4]

**9A**

(a) Using Fermat's principle, derive the second-order differential equation for the trajectory $y = \xi(x)$ of a light ray passing through a medium described by refractive index $n(x, y)$. Formulate this expression so that there are no derivatives in any denominators. [7]

(b) A designer needs to determine the shape $y(x)$ of a barrier to be built between the points $(x, y) = (0, 0)$ and $(1, 0)$. The designer has been told to maximise the area $A = \int_0^1 y\,dx$, but ensure that the cost

$$C = \int_0^1 \left( y + \left(\frac{dy}{dx}\right)^2 \right) dx$$

matches the budget $B$. Using calculus of variations, determine the optimal shape $y(x)$ and the area $A > 0$ enclosed. [13]

**10B**

The Sturm–Liouville eigenvalue equation is

$$-\left[p(x)\psi'(x)\right]' + q(x)\psi(x) = \lambda\, w(x)\, \psi(x)\,, \qquad (\star)$$

where $p(x) > 0$, $q(x) > 0$ and $w(x) > 0$ for $a \leqslant x \leqslant b$, and primes denote differentiation with respect to $x$.

(a) Show that for particular boundary conditions (which you must specify) finding the eigenvalues $\lambda$ in $(\star)$ is equivalent to finding the stationary values of the functional

$$\Lambda[\psi(x)] = \frac{\displaystyle\int_a^b \left[p(x)\,\psi'(x)^2 + q(x)\,\psi(x)^2\right] dx}{\displaystyle\int_a^b w(x)\psi(x)^2\, dx}\,.$$

[6]

(b) A general function $\tilde{\psi}$ can be written as

$$\tilde{\psi}(x) = \sum_{n=0}^{+\infty} a_n\, \psi_n(x)\,,$$

where $a_n$ are constants and $\psi_n$ $(n = 0, 1, 2, \ldots)$ are orthonormal eigenfunctions of $(\star)$ with ordered eigenvalues $(\lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \ldots)$. Show that

$$\Lambda[\tilde{\psi}(x)] = \frac{\lambda_0 + \sum_{n=1}^{+\infty} |b_n|^2\, \lambda_n}{1 + \sum_{n=1}^{+\infty} |b_n|^2}\,,$$

where $b_n = a_n/a_0$. Explain how this result allows estimation of the lowest eigenvalue $\lambda_0$. [6]

(c) Consider the particular case of the Mathieu equation

$$-\psi''(x) + \cos(\pi x)\,\psi(x) = \lambda\,\psi(x)\,,$$

for $0 \leqslant x \leqslant 1$ with the boundary conditions $\psi(0) = \psi(1) = 0$. Estimate the lowest eigenvalue $\lambda_0$ using the trial function $\tilde{\psi}(x) = \sin(\pi x) + \alpha\, \sin(2\pi x)$, with $\alpha \in \mathbb{R}$. [8]

## END OF PAPER