MATHEMATICS (1)

Before you begin read these instructions carefully:
You may submit answers to no more than six questions. All questions carry the same number of marks.
The approximate number of marks allocated to a part of a question is indicated in the right hand margin.
Write on one side of the paper only and begin each answer on a separate sheet.

At the end of the examination:
Each question has a number and a letter (for example, 6C).
Tie up each answer in a separate bundle, marked with the question number.
Do not join the bundles together.
For each bundle, a gold cover sheet must be completed and attached to the bundle.
A separate green master cover sheet listing all the questions attempted must also be completed.
Every cover sheet must bear your examination number and desk number.

Calculators are not permitted in this examination.

STATIONERY REQUIREMENTS
6 gold cover sheets and treasury tags
Green master cover sheet
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.
The interior of a paraboloidal body is defined parametrically in the Cartesian coordinate system \( x = (x, y, z) \) with

\[
\begin{align*}
x &= uv \cos \phi, \\
y &= uv \sin \phi, \\
z &= \frac{1}{2}(u^2 - v^2),
\end{align*}
\]

where \( 0 \leq u \leq v \), \( 0 \leq v < 1 \) and \( -\pi \leq \phi < \pi \).

(a) Sketch this body and describe its key characteristics.  

(b) Using \( (u, v, \phi) \) as a coordinate system, determine the Cartesian components of the vectors \( h_u, h_v \) and \( h_\phi \) such that the Cartesian differential \( dx \) is given by

\[
dx = h_u du + h_v dv + h_\phi d\phi.
\]

Determine also the corresponding scale factors. Is the coordinate system \( (u, v, \phi) \) orthogonal (you must justify your answer)?

(c) Determine the Jacobian for this coordinate transformation.

(d) Evaluate the integral

\[
I = \int_S F \cdot dS,
\]

where \( S \) is the surface of the paraboloidal body, \( dS \) is an element of vector area and \( F = \nabla \Omega \) with \( \Omega = x^3 + (x^2 + y^2)z \).
A linear wave with constant frequency $\omega$ can be described in a suitably rotated and rescaled orthogonal coordinate system $(x, y)$ by the pair of equations

$$\frac{\partial b}{\partial t} + \omega \frac{\partial \psi}{\partial y} + \epsilon \left( \frac{\partial \psi}{\partial x} - \tilde{\nabla}^2 b \right) = 0,$$

$$\frac{\partial}{\partial t} \tilde{\nabla}^2 \psi - \omega \frac{\partial b}{\partial y} - \epsilon \frac{\partial b}{\partial x} = 0,$$

where $b = b(x, y, t)$, $\psi = \psi(x, y, t)$, $\epsilon \ll 1$ is a constant parameter and

$$\tilde{\nabla}^2 = \epsilon^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

(a) Use the substitutions

$$b = (b_0 + \epsilon b_1 + \cdots) e^{-i\omega t},$$

$$\psi = (\psi_0 + \epsilon \psi_1 + \cdots) e^{-i\omega t},$$

to rewrite this system in the form

$$P + Q\epsilon + O(\epsilon^2) = 0,$$

$$R + S\epsilon + O(\epsilon^2) = 0.$$

Here, $b_i$ and $\psi_i$ (for $i = 0, 1, \ldots$) depend only on $x$ and $y$ and the functions $P$, $Q$, $R$ and $S$ may involve $b_i$, $\psi_i$ and/or derivatives of these. [5]

(b) Show that $P = R = 0$ if

$$b_0 = -i \frac{\partial \psi_0}{\partial y}.$$

[3]

(c) Set $P = Q = R = S = 0$ and eliminate $b_i$ to determine a differential equation for $\psi_0$. Show that this differential equation is satisfied when

$$\left( 2 \frac{\partial}{\partial x} + i \frac{\partial^3}{\partial y^3} \right) \psi_0 = f(x),$$

where $f(x)$ is an arbitrary function of $x$. [6]

(d) Assuming that $f(x) = 0$ and using separation of variables, find the solution to (\ref{derivative_equation}) for which $\psi_0(x = 0, y) = e^{iky}$ and $\psi_0 \to 0$ as $x \to \infty$, where $k$ is a real constant. [6]
Consider the second-order differential equation

\[
\frac{d^2 y(x)}{dx^2} + \frac{\alpha}{x} \frac{dy(x)}{dx} + \frac{(\alpha - 1)^2}{4x^2} y(x) = f(x),
\]

with \( \alpha \) a real constant.

(a) Find the general solution \( y(x) \) to (†) for the case \( f(x) = 0 \).

(b) Construct the Green’s function \( G(x, \xi) \) for (†) in the region \( x \geq 0 \) subject to the boundary conditions

\[
G(0, \xi) = \left. \frac{dG(x, \xi)}{dx} \right|_{x=0} = 0.
\]

(c) Use your Green’s function to solve (†) for the case \( f(x) = x \) in the region \( x \geq 0 \), subject to the boundary conditions

\[
y(0) = \left. \frac{dy(x)}{dx} \right|_{x=0} = 0,
\]

with \( \alpha > -5 \).
The Fourier transform \( \tilde{f}(k) \) of a function \( f(x) \) is given by
\[
\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx} \, dx.
\]

(a) Prove that the Fourier transform of the convolution
\[
h(x) = f * g := \int_{-\infty}^{\infty} f(t)g(x-t) \, dt
\]
is given by \( \tilde{h}(k) = \tilde{f}(k)\tilde{g}(k) \), where \( \tilde{f} \) and \( \tilde{g} \) are the Fourier transforms of \( f \) and \( g \), respectively. \[4\]

(b) Let
\[
f(x) = \begin{cases} 1, & |x| < \frac{1}{2}, \\ 0, & \text{otherwise}; \end{cases} \quad g(x) = \begin{cases} 1 - |x|, & |x| < 1, \\ 0, & \text{otherwise}. \end{cases}
\]
Show that
\[
\tilde{f}(k) = \frac{2}{k} \sin \frac{k}{2}, \quad \tilde{g}(k) = \frac{4}{k^2} \sin^2 \frac{k}{2},
\]
hence find the convolution of \( f \) with itself. \[10\]

(c) State Parseval’s identity and use the results from part (b) to evaluate
\[
\int_{-\infty}^{\infty} \frac{\sin^4 x}{x^4} \, dx.
\]
\[6\]
(a) What does it mean for an $n \times n$ matrix $A$ to be diagonalizable? Show that $A$ is diagonalizable if and only if it has $n$ linearly independent eigenvectors.

(b) Diagonalize the matrix

$$M = \begin{bmatrix} a & b - a & c - b \\ 0 & b & c - b \\ 0 & 0 & c \end{bmatrix},$$

where $a, b, c$ are arbitrary real numbers.

(c) Let $I$ be the identity matrix. Then, for a matrix $A$ such that $A^k \to 0$ as $k \to \infty$, where 0 is the zero matrix, the following equality is true: $(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$.

Find necessary and sufficient conditions on $a, b, c \in \mathbb{R}$ that ensure $M^k \to 0$ as $k \to \infty$, hence determine for such $M$ the entries of the matrix $(I - M)^{-1}$.

(d) For a set of $n$ linearly independent vectors $(x_i)_{i=1}^{n}$, let $B = (B_{i,j})_{i,j=1}^{n}$ be the matrix with the entries

$$B_{i,j} = x_i^T x_j.$$

Prove that the quadratic form associated with the matrix $B$,

$$Q(c) := c^T B c,$$

is positive definite, that is to say, $Q(c) > 0$ for all non-zero $c \in \mathbb{R}^n$. 

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Natural Sciences IB, Mathematics Paper 1
6C

(a) Define a skew-Hermitian matrix and show that its eigenvalues are purely imaginary. Define a unitary matrix and show that its eigenvalues have modulus 1.

(b) Show that if $A$ is skew-Hermitian with distinct eigenvalues $(\lambda_i)_{i=1}^n$, then its eigenvectors are orthogonal and
$$ A = U D U^\dagger, $$
where $D = \text{diag} (\lambda_1, \ldots, \lambda_n)$ and $U$ is unitary.

(c) Let $\exp(A)$ be the matrix exponent,
$$ \exp(A) := \sum_{k=0}^\infty \frac{A^k}{k!}. $$
Using (b) or otherwise, show that if $A$ is skew-Hermitian with distinct eigenvalues, then $U = \exp(A)$ is unitary.

(d) Suppose that a unitary matrix $U$ can be written as $U = A + iB$, where $A$ and $B$ are real antisymmetric matrices, each with $n$ distinct eigenvalues. Show that $A$ and $B$ have the same eigenvectors and determine the eigenvalues of $A$ and $B$ in terms of eigenvalues of $U$.

7C

(a) Write down the Cauchy-Riemann equations for an analytic function $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, hence show that curves of constant $u$ and curves of constant $v$ intersect at right angles.

(b) Show that if $g(z)$ is analytic and $|g(z)|$ is constant, then $g$ is constant.

(c) For $|z| < \infty$, find and classify the singularities of the following functions:

(i) $\cot z$, (ii) $\cot \frac{1}{z}$, (iii) $e^{\cot z}$.

(d) Find the power series expansion (with real coefficients) about $z = 1$ of the function
$$ f(z) = \frac{2z}{z^2 + 1}. $$
Determine the radius of convergence of this series.
8B

(a) Define an ordinary point and a regular singular point of the ordinary differential equation
\[ \frac{d^2 y}{dx^2} + p(x) \frac{dy}{dx} + q(x)y = 0. \]
What are the implications for the existence of a series solution at such points? [6]

(b) Consider the differential equation
\[ \frac{d}{dx} \left[ (1 - x) \frac{dy}{dx} \right] + [c^2 (1 - x) + \lambda] y = 0, \]
with \( c > 0 \) and \( \lambda \) real.

(i) Identify the singular point(s) of (†). [2]

(ii) Show that a series solution of (†) is
\[ y_1 = (1 - x)^\sigma \sum_{n=0}^{+\infty} a_n (1 - x)^n, \]
where you should determine \( \sigma \) and find the recurrence relation for \( a_n \). Why can only one solution of this form be found? [6]

(iii) Give the general solution of (†) in terms of an explicit integral involving \( y_1 \) and the Wronskian. [2]

(iv) Hence show that any solution of (†) that is linearly independent of \( y_1 \) must behave like a logarithm of \( 1 - x \) near \( x = 1 \). [4]

9A

(a) Using Fermat’s principle, derive the second-order differential equation for the trajectory \( y = \xi(x) \) of a light ray passing through a medium described by refractive index \( n(x, y) \). Formulate this expression so that there are no derivatives in any denominators. [7]

(b) A designer needs to determine the shape \( y(x) \) of a barrier to be built between the points \( (x, y) = (0, 0) \) and \( (1, 0) \). The designer has been told to maximise the area \( A = \int_0^1 y \, dx \), but ensure that the cost
\[ C = \int_0^1 \left( y + \left( \frac{dy}{dx} \right)^2 \right) \, dx \]
matches the budget \( B \). Using calculus of variations, determine the optimal shape \( y(x) \) and the area \( A > 0 \) enclosed. [13]
The Sturm–Liouville eigenvalue equation is
\[- [p(x) \psi'(x)]' + q(x) \psi(x) = \lambda w(x) \psi(x), \quad (\star)\]
where $p(x) > 0$, $q(x) > 0$ and $w(x) > 0$ for $a \leq x \leq b$, and primes denote differentiation with respect to $x$.

(a) Show that for particular boundary conditions (which you must specify) finding the eigenvalues $\lambda$ in $(\star)$ is equivalent to finding the stationary values of the functional

$$\Lambda[\psi(x)] = \frac{\int_a^b \left[ p(x) \psi'(x)^2 + q(x) \psi(x)^2 \right] dx}{\int_a^b w(x) \psi(x)^2 dx}.$$  

(b) A general function $\tilde{\psi}$ can be written as

$$\tilde{\psi}(x) = \sum_{n=0}^{+\infty} a_n \psi_n(x),$$

where $a_n$ are constants and $\psi_n$ ($n = 0, 1, 2, \ldots$) are orthonormal eigenfunctions of $(\star)$ with ordered eigenvalues ($\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$). Show that

$$\Lambda[\tilde{\psi}(x)] = \frac{\lambda_0 + \sum_{n=1}^{+\infty} |b_n|^2 \lambda_n}{1 + \sum_{n=1}^{+\infty} |b_n|^2},$$

where $b_n = a_n/a_0$. Explain how this result allows estimation of the lowest eigenvalue $\lambda_0$.  

(c) Consider the particular case of the Mathieu equation

$$-\psi''(x) + \cos(\pi x) \psi(x) = \lambda \psi(x),$$

for $0 \leq x \leq 1$ with the boundary conditions $\psi(0) = \psi(1) = 0$. Estimate the lowest eigenvalue $\lambda_0$ using the trial function $\tilde{\psi}(x) = \sin(\pi x) + \alpha \sin(2\pi x)$, with $\alpha \in \mathbb{R}$.  

END OF PAPER

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