UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Wednesday, 15 June, 2022 9:00 am to 12:00 pm

MATHEMATICS (2)

This is a closed book exam.

Before you begin read these instructions carefully:

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

You may submit answers to ***all*** *of section A, and to no more than* ***five*** *questions from section B.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on one side of the paper only and begin each answer on a separate sheet. *(For this purpose, your section A attempts should be considered as one single answer.)*

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Tie up ***all of your section A answer*** *in a single bundle, with a completed gold cover sheet.*

Tie up ***each*** *section B answer in a* ***separate*** *bundle, marked with the question number.* ***Do not join the bundles together****. For each bundle, a gold cover sheet* ***must*** *be completed and attached to the bundle.*

A ***separate*** *green master cover sheet listing all the questions attempted* ***must*** *also be completed. (Your section A answer may be recorded just as A: there is no need to list each individual short question.)*

Every cover sheet must bear your blind grade number and desk number, and should not include your name.

STATIONERY REQUIREMENTS	***SPECIAL REQUIREMENTS***
6 gold cover sheets and treasury tags	*No calculators may be used.*
Green master cover sheet	*No electronic devices may be used.*
Script paper	

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1

Calculate

(a) $(1,2,3)\times(3,2,1)$, [1]

(b) $(1,2,0)\times[(1,-1,0)\times(0,0,1)]$. [1]

2 Re-express the following as matrix equations (that is, without subscripts)

(a) $C_{ik}=B_{jk}A_{ij}$, [1]

(b) $t=A_{ii}$. [1]

3 Is the following ordinary differential equation (a) separable, (b) homogeneous and (c) linear?

$$\frac{dy}{dx}=x^2-xy.$$

[2]

4 Does the following system of simultaneous equations have non-zero solutions?

$$\begin{pmatrix}1&1&1\\1&1&2\\1&2&3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

[2]

5 Consider the partial differential equation

$$4\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}=0.$$

Find the values of λ such that $u=f(x+\lambda y)$ for arbitrary function f is a solution. [2]

6 Write down the Fourier series on $0\leqslant x<2\pi$ for the function

$$f(x)=\sin 3x\cos x.$$

[2]

7 State the divergence theorem for vector field $\mathbf{F}(\mathbf{r})$ and closed surface S. [2]

8 Calculate the first two non-zero terms in the Maclaurin series for

$$f(x) = \sin 3x \cos x.$$

[2]

9 Find the indefinite integral of the function $x \ln x$ for $x > 0$. [2]

10 Find the four second order partial derivatives of the function

$$f(x, y) = x^2 y^3.$$

[2]

SECTION B

11Z

(a) By using the scalar triple product, give a condition for vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ to form a basis in three-dimensional vector space. [2]

Assuming that $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ do form a basis, an arbitrary vector $\boldsymbol{y}$ can be written as

$$\boldsymbol{y} = \alpha\boldsymbol{u} + \beta\boldsymbol{v} + \gamma\boldsymbol{w}.$$

Find the coefficients α, β and γ in terms of scalar triple products involving vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ and $\boldsymbol{y}$. [3]

(b) A vector $\boldsymbol{x} \in \mathbb{R}^3$ satisfies the equation

$$\boldsymbol{x} = \boldsymbol{a} + (\boldsymbol{b} \cdot \boldsymbol{x})\boldsymbol{c}, \qquad (\dagger)$$

where $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ are three non-zero position vectors in $\mathbb{R}^3$.

(i) Take the cross product of ($\dagger$) with vector $\boldsymbol{c}$ and thus deduce an expression for $\boldsymbol{x}$ in terms of the vectors $\boldsymbol{a}$ and $\boldsymbol{c}$, and a parameter λ. [4]

(ii) To determine λ, substitute the expression for $\boldsymbol{x}$ you found in (b)(i) into ($\dagger$) and thus find conditions on $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ for which λ has a unique value, has multiple values or is undefined. [7]

(iii) Solve ($\dagger$) for $\boldsymbol{x}$ in the cases where solutions exist, and interpret these solutions geometrically. [4]

12W

Consider the function

$$f(x, y) = xy(x^2 + y^2 - 1)\,.$$

(a) Calculate ∇f. [2]

(b) Calculate the directional derivative of f at point A given by the position vector $\boldsymbol{a} = (\frac{1}{2}, \frac{1}{4})$ in the direction of position vector $\boldsymbol{a}$. [2]

(c) Find all the stationary points of $f(x, y)$ and classify them, providing your reasoning for classification. [8]

(d) Sketch the contour lines of $f(x, y)$ in the region $-1.5 < x < 1.5,\ -1.5 < y < 1.5$. [4]

(e) Add arrows to the sketch with the contour lines to show the direction of ∇f near the stationary points and on the contour lines corresponding to $f(x, y) = 0$. [4]

13Z

(a) A two-dimensional vector field is given in Cartesian coordinates by

$$\boldsymbol{F} = (xy, x^2 + y).$$

Calculate, by using any method, the line integrals of $\boldsymbol{F}$ along the following curves, each parameterised by $t \in [0, 1]$:

(i) (t, t^2), [2]

(ii) (t^2, t), [2]

(iii) $(2t - t^2, t^4 - 4t^3 + 4t^2)$. [6]

(b) Two three-dimensional vector fields are given by

$$\boldsymbol{G} = \boldsymbol{a} \times \boldsymbol{r}$$

and

$$\boldsymbol{H} = f(|\boldsymbol{r}|)\boldsymbol{r},$$

where $\boldsymbol{r}$ denotes the position in $\mathbb{R}^3$, $\boldsymbol{a}$ is a fixed vector, and f is an arbitrary function. Explain whether each of $\boldsymbol{G}$ and $\boldsymbol{H}$ can be written as the gradient of a scalar function and, if they can, find all such functions. You may assume that any needed derivatives and integrals of f exist. [10]

14V

An enthusiast planted 5 avocado seeds and 5 lemon seeds, hoping to grow some fruit trees. Let A denote the event that a given avocado seed grows into a tree, and L denote the event that a given lemon seed grows into a tree within the growing season. Let $P(A) = x$ and $P(L) = y$. Assume that seeds grow into trees with independent probabilities. Throughout this question you may leave your answers in terms of x, $1 - x$, y and $1 - y$.

(a) What is the probability that

(i) exactly two lemon seeds grow into trees? [2]

(ii) at least two lemon seeds grow into trees? [2]

(iii) at least two seeds grow into trees? [4]

(b) Given that exactly 2 trees grew from the 10 seeds, what is the probability that they are both of same kind (both avocados or both lemons)? [5]

(c) In a season, mature avocado trees independently produce a number of seeds that are uniformly distributed between 30 and 39 (inclusive). What is the probability of getting exactly 62 seeds in a season, given two mature avocado trees? [2]

(d) Find an expression for the number of avocado seeds that should be planted to have a 95% chance of growing at least one new tree. [5]

15Y

(a) Using an integrating factor or other method, find the general solution of

$$(2\sin y - x)\frac{dy}{dx} = \tan y.$$

[6]

Hence determine the solution in each of the following cases;

(i)

$$y(0) = 0,$$

[2]

(ii)

$$y(\tfrac{1}{2}) = \frac{\pi}{6},$$

[2]

(iii)

$$y(\tfrac{1}{2}) = \frac{5\pi}{6}.$$

[2]

(b) Using a complementary function or other method, find the solution of

$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = 6,$$

subject to the boundary conditions

$$\begin{aligned} y(0) &= 0, \\ y(1/3) &= \frac{1-e^{-1}}{2}. \end{aligned}$$

[6]

Determine the value of y(1). [2]

16R

A closed shape is made of a cone ($z \geqslant 0$) and paraboloid ($z < 0$). The expressions for the elementary surface vector area, $d\boldsymbol{S}$, along outward normal on the surface of this closed shape are given by the following expressions:

$$d\boldsymbol{S} = \alpha \left(\frac{\partial \boldsymbol{r}}{\partial x} \times \frac{\partial \boldsymbol{r}}{\partial y} \right) dx\, dy$$

$$\text{where } \begin{cases} \alpha = 1 \quad \text{and} \quad \boldsymbol{r} = (x, y, 1 - \sqrt{x^2 + y^2})^\top & \text{for } \; z \geqslant 0, \\ \alpha = -1 \quad \text{and} \quad \boldsymbol{r} = (x, y, -1 + x^2 + y^2)^\top & \text{for } \; z < 0. \end{cases}$$

(a) Sketch the closed shape and find the overall ranges which the values x-, y- and z-coordinates of the points on the surface of this shape can take. [4]

(b) Find the flux, $\Phi = \iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$, of the vector field $\boldsymbol{F} = (x^{2k}y, xy^{2k}, z^k)^\top$ through this closed shape by direct evaluation of the surface integrals and without using the divergence theorem in the following cases:

(i) $k = 0$, [7]

(ii) $k = 1$. [9]

17S

(a) Determine if the matrix $\boldsymbol{A}$ is orthogonal, where

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -3 & 1 \\ -1 & 0 & 1 \end{pmatrix}.$$

[2]

By using suffix notation, show that if $\boldsymbol{A}$ and $\boldsymbol{B}$ are two $(n \times n)$ $(n \in \mathbb{N})$ orthogonal matrices then $\boldsymbol{AB}$ is orthogonal. [6]

(b) Consider the following set of simultaneous linear equations, in which a is a real constant

$$\begin{aligned} x + 2y + 3z \; &= 1, \\ x + y + (a-1)z \; &= 2, \\ ax + 2y + 3z \; &= 3. \end{aligned}$$

(i) Express this set of linear equations in matrix form $\boldsymbol{Ax} = \boldsymbol{b}$. [2]

(ii) Calculate $\det(\boldsymbol{A})$ as a function of a. For which values of a does the determinant vanish? [4]

(iii) Classify the number of solutions of these linear equations for all values of a. [6]

18W

(a) The set of functions, $\{e_1(x), e_2(x), \ldots\}$, forms a complete set of orthogonal functions for $-\pi \leqslant x \leqslant \pi$. Hence, an integrable function, $f(x)$, can be expanded on this interval as

$$f(x) = \sum_{n=1}^{\infty} \alpha_n e_n(x)\,.$$

(i) Give an example of $\{e_1(x), e_2(x), \ldots\}$ in which $e_n(x)$ are represented by trigonometric functions. [1]

(ii) In your example, what is meant by the orthogonality of two basis functions $e_n(x)$ and $e_m(x)(m \neq n)$? [3]

(iii) For your example, rewrite the expansion of $f(x)$ by using the standard notations for the coefficients, i.e. a_n and b_n, and give the expressions for these coefficients? [3]

(b) Find the Fourier series expansion of the 2π-periodic function which is defined for $-\pi \leqslant x < \pi$ as follows,

$$f(x) = \sin(x) - 2\sin(2x) + |x|(|x| - \pi)\,.$$

[10]

(c) Use the Fourier series found in (b) to demonstrate that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}\,.$$

[3]

19X*

(a) Explain how Lagrange multipliers are used to find the extreme values of a function $f(x, y, z)$ subject to a constraint $g(x, y, z) = A$ where A is constant. [5]

(b) The temperature T of a point on the unit sphere is given by

$$T(x, y, z) = xy + yz.$$

Find the temperatures and locations of the hottest and coldest points on the sphere, and of any other points where the temperature gradient is zero on the surface of the sphere. [12]

(c) Explain, without solving the equations, how the calculation would be modified if an additional constraint, $y = h(x)$ were imposed. [3]

20T*

(a) The function $T(x,y)$ satisfies the following diffusion equation

$$a\frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial y^2}, \qquad (\dagger)$$

on the interval $0 \leqslant y \leqslant 1$ and $x \geqslant 0$ subject to the boundary conditions that

$$\begin{aligned} T(x,0) &= T(x,1) = 0, \\ T(0,y) &= y. \end{aligned}$$

By using the separable function $T(x,y) = f(x)g(y)$, show that equation ($\dagger$) may be written as

$$a\frac{1}{f}\frac{df}{dx} = \frac{1}{g}\frac{d^2 g}{dy^2} = -\lambda^2,$$

for constant λ. [2]

Determine all the functions $f(x)$ and $g(y)$ satisfying these equations and the homogeneous boundary conditions. Hence determine the solution to ($\dagger$) with the full set of boundary conditions. [10]

(b) Show that the function

$$h(x,t) = at^{-1/3}\left(1 - \frac{x^2}{t^{2/3}}\right) - bt$$

satisfies the partial differential equation

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) - 1, \qquad (\ddagger)$$

so long as the constants a and b take particular values. [5]

Determine the value of the constants a and b, and hence write the general form of the solution to equation ($\ddagger$). [3]

END OF PAPER