UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB

Friday, 11 June, 2021 1:00pm to 4:00pm

MATHEMATICS (2)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, ***6B****).*

Answers must be tied up in ***separate*** *bundles, marked* ***A, B or C*** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet ***must*** *be completed and attached to the bundle.*

A ***separate*** *green master cover sheet listing all the questions attempted* ***must*** *also be completed.*

Every cover sheet must bear your examination number and desk number.

Calculators are not permitted in this examination.

STATIONERY REQUIREMENTS
3 blue cover sheets and treasury tags
Green master cover sheet
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1A

(a) The inner product of two functions $f(x)$ and $g(x)$, defined on the closed interval $a \leqslant x \leqslant b$, is

$$\langle f|g\rangle = \int_a^b f^*(x)\, g(x)\, w(x)\, \mathrm{d}x$$

where $w(x) \geqslant 0$. Consider the operator

$$\mathcal{L} = -\frac{1}{w(x)}\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) - q(x)\right] \quad \text{with} \quad a \leqslant x \leqslant b\,, \qquad (\star)$$

and where $p(x)$, $q(x)$, $w(x)$ are real functions with $p(x) > 0$.

(i) Derive the boundary conditions under which $\mathcal{L}$ is self-adjoint over the range $a \leqslant x \leqslant b$, with respect to the inner product defined above. [3]

(ii) Show that the eigenvalues of $\mathcal{L}$ are real and that any two eigenfunctions of $\mathcal{L}$ with distinct eigenvalues are orthogonal. [4]

(iii) Derive the solution to the inhomogeneous equation

$$\mathcal{L}y = f$$

as an eigenfunction expansion assuming that $y(x)$ and $f(x)$ satisfy the boundary conditions derived in part (i). You may assume that $\mathcal{L}$ does not have zero eigenvalues. [3]

(b) Consider the eigenvalue problem

$$\mathcal{L}y \equiv -\frac{1}{x^4}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{2}{x^5}\frac{\mathrm{d}y}{\mathrm{d}x} - a\,y = \lambda\,y\,, \qquad (\dagger)$$

where a is a positive constant and $y(0) = y(1) = 0$ with $0 \leqslant x \leqslant 1$.

(i) Show that $\mathcal{L}$ in ($\dagger$) can be written as in ($\star$) and identify the functions $p(x)$, $q(x)$ and $w(x)$. [4]

(ii) Find the eigenvalues and orthonormal eigenfunctions of $\mathcal{L}$.

[*Hint: Consider the substitution* $z = \frac{x^3}{3}$.] [6]

2A

(a) Consider solutions to the three dimensional Laplace equation,

$$\nabla^2 \Psi(\mathbf{r}) = 0\,,$$

in a volume V with boundary surface S. Assume that $\Psi(\mathbf{r})$ satisfies a Dirichlet boundary condition $\Psi(\mathbf{r}) = f(\mathbf{r})$ on S, with f defined on S. Using the divergence theorem show that the solution $\Psi(\mathbf{r})$ is unique. [10]

(b) The function $\Phi(x, y)$ defined on the square domain $0 \leqslant x \leqslant \pi$, $0 \leqslant y \leqslant \pi$, satisfies

$$\nabla^2 \Phi(x, y) + \left(m^2 + p^2\right) \Phi(x, y) = 0\,,$$

where ∇^2 is the two dimensional Laplace operator and m and p are integers. Let S denote the boundary of the integration domain with unit normal $\mathbf{n}$.

For each of the following, using separation of variables find one non-zero solution for $\Phi(x, y)$ such that:

(i) $\Phi(x, y)$ vanishes on S. [6]

(ii) $\mathbf{n} \cdot \nabla \Phi(x, y)$ vanishes on S. [4]

3A

(a) The modified Bessel function of the second kind $K_m(r)$ satisfies the differential equation

$$r^2\frac{\mathrm{d}^2 K_m(kr)}{\mathrm{d}r^2} + r\frac{\mathrm{d}K_m(kr)}{\mathrm{d}r} - (m^2 + k^2r^2)K_m(r) = 0\,.$$

In three dimensions the function $\Phi(\mathbf{r})$ is defined as

$$\Phi(\mathbf{r}) = Ar^{-p}K_m(kr) \quad ,$$

where $r = |\mathbf{r}|$ and A, k, p and m are real constants. By considering the region $r \neq 0$, determine the values of p and m for which $\Phi(\mathbf{r})$ satisfies the equation

$$\nabla^2\Phi(\mathbf{r}) - k^2\Phi(\mathbf{r}) = \delta^3(\mathbf{r}) \quad , \qquad (\dagger)$$

where in spherical polar coordinates

$$\nabla^2\Phi(\mathbf{r}) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi(\mathbf{r})}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi(\mathbf{r})}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi(\mathbf{r})}{\partial^2\phi} \quad .$$ [10]

(b) The functional form of this solution is

$$\Phi(\mathbf{r}) = A\sqrt{\frac{\pi}{2k}}\frac{e^{-kr}}{r}\,.$$

Using the divergence theorem and the behaviour of equation ($\dagger$) near the origin determine the value of A.

[*Hint: Consider integrating equation* ($\dagger$) *over a small region containing the origin.*] [6]

(c) Now consider the equation

$$\nabla^2\Psi(\mathbf{r},\mathbf{r}') - k^2\Psi(\mathbf{r},\mathbf{r}') = \delta^3(\mathbf{r}-\mathbf{r}')\,.$$

Using the method of images determine the Green's function $\Psi(\mathbf{r},\mathbf{r}')$ for a domain D with Dirichlet boundary conditions on $z = 0$ and where D is the half-space of $\mathbb{R}^3$ with $z \geqslant 0$. [4]

4A

(a) State and prove Cauchy's theorem. *[You may assume the Cauchy-Riemann equations.]* [4]

(b) Use contour integration to determine the value of

$$\int_0^{+\infty} \frac{x^{1/n}}{a^2 + x^2} \mathrm{d}x$$

where a is real and positive, and $n > 1$. State clearly the location of any branch cut required. [8]

(c) By applying the calculus of residues show that for $0 < a < 1$

$$\int_0^{2\pi} \frac{1}{a^2 + \tan^2 \theta} \, \mathrm{d}\theta = \frac{C}{a(1+a)} \,,$$

where C is a constant you should determine.

[Hint: You may use

$$\operatorname*{res}_{z=\pm\sqrt{\frac{1-a}{1+a}}} f(z) = \frac{1}{2(1-a^2)a} \,,$$

with

$$f(z) = \frac{1}{z} \frac{(1+z^2)^2}{a^2(1+z^2)^2 - (1-z^2)^2} \,.$$

] [8]

5B

(a) The Fourier transform and its inverse for a function $f(t)$ are given by

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\mathrm{d}t\,, \qquad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)e^{i\omega t}\mathrm{d}\omega\,.$$

(i) Compute the Fourier transforms of the functions $f_1(t) = e^{i\omega_0 t}$ and $f_2(t) = \sin(\omega_0 t)$, where ω_0 is a real constant. [2]

(ii) Compute the inverse Fourier transform of the convolution $(\tilde{f} * \tilde{g})(\omega)$ in terms of $f(t)$ and $g(t)$. [2]

(iii) Show that the Fourier transform of the Heaviside function

$$H(t) = \begin{cases} 1 & \text{for } \; t > 0 \\ 0 & \text{for } \; t < 0 \end{cases}\,,$$

is $\tilde{H}(\omega) = \frac{A}{\omega} + B\delta(\omega)$, where δ denotes the Dirac delta function, and A and B are constants you should determine.

[Hint: Compute the inverse Fourier transform of $\frac{1}{\omega}$. You may use $\int_0^\infty \frac{\sin x}{x}\mathrm{d}x = \frac{\pi}{2}$.] [4]

(iv) Compute the Fourier transform of $f(t) = H(t)\sin(\omega_0 t)$. [3]

(b) Consider the differential equation

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} + F(t)\delta(x - x_0)\,, \qquad (\dagger)$$

for the function $y(t,x)$ on the domain $0 \leqslant x \leqslant L$, $-\infty < t < \infty$, with boundary conditions $y(t,0) = y(t,L) = 0$, and where x_0 is a constant with $0 < x_0 < L$.

(i) Apply a Fourier transform in the time domain, treating x as a constant parameter in this process, to show that equation $(\dagger)$ can be rewritten as

$$\frac{\partial^2 \tilde{y}(\omega, x)}{\partial x^2} = -\omega^2 \tilde{y}(\omega, x) - \tilde{F}(\omega)\delta(x - x_0)\,,$$

where you may assume that the individual Fourier transforms $\tilde{y}$ and $\tilde{F}$ exist. [2]

(ii) Define $b_n(\omega) = \frac{2}{L}\int_0^L \tilde{y}(\omega, x)\sin(k_n x)\mathrm{d}x$ for positive integer n and $k_n = \frac{n\pi}{L}$. Show that

$$b_n(\omega) = \alpha_n \frac{\tilde{F}(\omega)}{k_n^2 - \omega^2}\,,$$

where α_n is a constant (depending on n) which you should determine. [4]

[QUESTION CONTINUES ON THE NEXT PAGE]

(iii) Use the coefficient functions $b_n(\omega)$ to show that

$$y(t,x) = C\sum_{n=1}^{\infty}\sin(k_n x_0)\,\sin(k_n x)\int_{-\infty}^{\infty}\frac{\tilde{F}(\omega)e^{i\omega t}}{k_n^2-\omega^2}\mathrm{d}\omega\,,$$

is a solution to equation (†), where C is a constant that you should determine. [3]

6B

(a) Let $\{\boldsymbol{e}_i\}$ and $\{\boldsymbol{e}'_i\}$ with $i = 1, 2, 3$ denote two sets of orthonormal basis vectors.

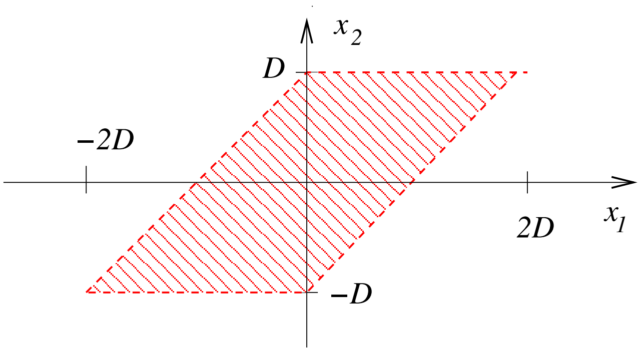
(i) Write down the transformation law for a tensor $S_{a_1 a_2 \ldots a_n}$ of order n under an orthogonal basis transformation

$$\boldsymbol{e}'_i = L_{ij}\boldsymbol{e}_j \, .$$

What are the possible values of the determinant of the transformation matrix, $\det \mathbf{L}$? [2]

(ii) Let ϵ_{ijk} denote the Levi-Civita symbol. Use the transformation law for pseudo-tensors to show that under an orthogonal transformation, $\epsilon'_{ijk} = \epsilon_{ijk}$. [5]

(b) Consider an infinitesimally thin mass distribution of parallelogram shape in the (x_1, x_2) plane defined by the corner points $(-2D, -D)$, $(0, D)$, $(2D, D)$ and $(0, -D)$, as shown in the figure. The mass density is $\rho(x_1, x_2, x_3) = \sigma\delta(x_3)$, D and σ are positive real constants, and δ denotes the Dirac delta function.

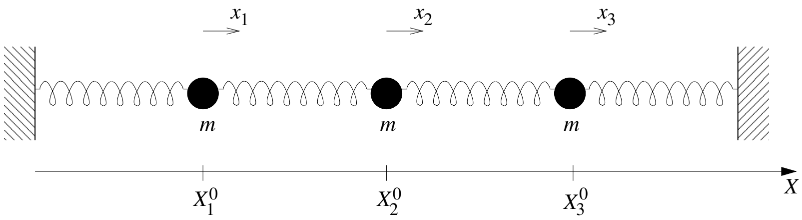


(i) Calculate the inertia tensor $I_{ij} = \int (x_k x_k \delta_{ij} - x_i x_j)\, \rho \, \mathrm{d}V$ of this mass distribution. [8]

(ii) Determine the moments of inertia and the principal axes of this mass distribution. [5]

7B

(a) Consider the one-dimensional motion of three equal point masses m connected to each other and fixed walls at either end by four equal springs as shown in the figure.



The masses, springs and walls are all aligned along the X axis, and the masses have equilibrium positions X_1^0, X_2^0 and X_3^0. The masses can move along the X direction with displacements $x_i(t) = X_i(t) - X_i^0$ and each spring responds to a change in length Δx with a restoring force $F = -k\Delta x$ where k is a positive constant.

(i) Write down the Lagrangian in the form $\mathcal{L} = \frac{1}{2}T_{ij}\dot{x}_i\dot{x}_j - \frac{1}{2}V_{ij}x_ix_j$ for this system, where summation over i and j is assumed. Be sure to give explicit expressions for the matrices $\mathbf{T}$ and $\mathbf{V}$. [3]

(ii) Determine the three normal mode frequencies in terms of k and m by solving the characteristic equation $\det(\mathbf{V} - \omega^2\mathbf{T}) = 0$. Determine the generalized eigenvectors associated with these normal mode frequencies. [4]

(iii) Write down the general solution for the motion of the masses in terms of the eigenvectors and normal mode frequencies. Specify explicitly the free parameters of this general solution. [2]

(b) Consider the same arrangement as in part (a) but now with N equal masses m connected to each other and the two fixed walls by $N + 1$ springs with spring constant k. The equilibrium positions of the masses are X_i^0, $i = 1, \ldots, N$.

(i) Write down the Lagrangian for this system and the resulting matrix $\mathbf{V} - \omega^2\mathbf{T}$. [2]

(ii) Show that the determinant $D_N = \det(\mathbf{V} - \omega^2\mathbf{T})$ can be evaluated from a recurrence relation for D_N in terms of D_{N-1} and D_{N-2}. [2]

(iii) Using the ansatz $D_N = \beta^N$, show that the recurrence relation is solved by $D_N = c_1\beta_1^N + c_2\beta_2^N$, where $\beta_1 + \beta_2 = 2k - m\omega^2$ and $\beta_1\beta_2 = k^2$. Use the expressions for D_N for small N to determine the coefficients c_1 and c_2 in terms of β_1 and β_2. *[Hint: Show that we can set $D_0 = 1$.]* [4]

(iv) Using this expression for D_N, show that the characteristic equation $D_N = 0$ leads to the condition $(\beta_1/\beta_2)^{N+1} = 1$. Based on your derivation, briefly explain how many distinct solutions to the characteristic equation you would expect for a given N. *[You do not need to compute these solutions.]* [3]

8C Let G be the set of matrices of the form

$$\begin{pmatrix} \cosh x & -\sinh x \\ -\sinh x & \cosh x \end{pmatrix},$$

where x is real.

(a) Show that G forms a group under matrix multiplication. [4]

(b) Define the terms *coset*, *normal subgroup* and *quotient group*, showing that the latter is indeed a group. [5]

(c) Let G' be the group generated by elements of G and the matrices

$$\mathbf{T} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathbf{P} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Show that G is a normal subgroup of G'. Identify the quotient group G'/G and give its standard name, justifying your answers. [7]

(d) Is the set of elements generated by $\mathbf{T}$ and $\mathbf{P}$ also a normal subgroup of G'? Justify your answer. [4]

9C

The dihedral group D_n, i.e. the group of symmetries of a regular polygon with n sides (for $n \geqslant 3$), is generated by elements R and m, where R is of order n, $m^2 = I$ and $Rm = mR^{-1}$.

(a) Derive the full set of elements of the group, showing that your answer is complete and that no elements are listed twice. Describe the actions of R and m geometrically. [5]

(b) Obtain the conjugacy classes of D_n, distinguishing the cases of odd and even n. [8]

(c) List all of the proper normal subgroups of D_n in the case when n is prime (and therefore odd). Justify that your list is complete. [7]

10C

In parts (c), (d) and (e) of this question you may quote without proof any general results from the lectures, provided these are clearly stated.

(a) Define a *representation*, *faithful representation* and *irreducible representation* (*irrep*) of a group G. [3]

(b) If $\mathbf{D}$ is a representation of a group G, prove that $\mathbf{D}(g^{-1}) = [\mathbf{D}(g)]^{-1}$ for any $g \in G$. [3]

(c) Consider the nonabelian group $H = \{1, -1, +i, -i, +j, -j, +k, -k\}$ with $i^2 = j^2 = k^2 = ijk = -1$. Find all the conjugacy classes of H. Deduce the number of inequivalent irreps and state their dimensions. [5]

(d) Given that the one-dimensional irreps of H have character $= 1$ for the group elements ± 1, construct the character table for H. [5]

(e) Consider the 4-dimensional representation $\mathbf{D}$ of H given by,

$$\mathbf{D}(1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{D}(-1) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

$$\mathbf{D}(i) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \mathbf{D}(j) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{D}(k) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Find the multiplicity with which each irrep appears in the decomposition of $\mathbf{D}$ into irreps. [4]

END OF PAPER