UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB

Wednesday, 9 June, 2021 11:00am to 2:00pm

MATHEMATICS (1)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, ***6A****).*

Answers must be tied up in ***separate*** *bundles, marked* ***A, B or C*** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet ***must*** *be completed and attached to the bundle.*

A ***separate*** *green master cover sheet listing all the questions attempted* ***must*** *also be completed.*

Every cover sheet must bear your examination number and desk number.

Calculators are not permitted in this examination.

STATIONERY REQUIREMENTS
3 blue cover sheets and treasury tags
Green master cover sheet
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1C

(a) For a vector field $\mathbf{A}$ in three dimensions, show that

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \nabla^2 \mathbf{A}\,.$$ [2]

(b) State the *divergence theorem*, taking care to define all the quantities which appear. [2]

(c) Oblate spheroidal coordinates (ρ, u, v) are related to Cartesian coordinates (x, y, z) by

$$\begin{aligned} x &= \cosh\rho\,\cos u\,\cos v\,, \\ y &= \cosh\rho\,\cos u\,\sin v\,, \\ z &= \sinh\rho\,\sin u\,, \end{aligned}$$

where $\rho \geqslant 0$, $-\pi/2 \leqslant u \leqslant \pi/2$ and $0 \leqslant v < 2\pi$. Show that these coordinates are orthogonal. Show that the volume element can be expressed as

$$\mathrm{d}V = \cosh\rho\,\cos u\,(\sinh^2\rho + \sin^2 u)\,\mathrm{d}\rho\,\mathrm{d}u\,\mathrm{d}v\,.$$ [5]

(d) Describe the surfaces of constant ρ, the surfaces of constant u and the surfaces of constant v. [4]

(e) Calculate the integral

$$\int_S \mathbf{A} \cdot \mathbf{n}\,\mathrm{d}S\,,$$

where $\mathbf{A} = (x\,y^2,\ y^2,\ 2)$ in Cartesian coordinates, S is the open surface $\rho = R =$ constant with $z \geqslant 0$, and $\mathbf{n}$ is the outward pointing unit normal to the surface. [7]

2A

Consider the three-dimensional heat equation for a temperature field $\psi(t,r,\theta)$ in an axially symmetric configuration

$$\frac{1}{\kappa}\frac{\partial\psi}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) ,$$

where $\kappa > 0$ is a constant, $r \geqslant 0$ and $0 \leqslant \theta \leqslant \pi$.

(a) By using separation of variables, write down ordinary differential equations for the radial, angular and temporal dependence of ψ. [8]

(b) Consider the spherically symmetric case for which $\psi(t,r)$ is a function of t and r only. Assuming that

$$r\,\psi(0,r) = \sin\left[\frac{\pi\,(r_+ - r)}{r_+ - r_-}\right] + \sin\left[\frac{2\pi\,(r_+ - r)}{r_+ - r_-}\right] \quad \text{and} \quad \psi(t,r_\pm) = 0\,,$$

with $r_+ > r_- > 0$, determine $\psi(t,r)$ in the interval $r_- \leqslant r \leqslant r_+$. [12]

3B

Consider the linear second order differential equation

$$\mathcal{L}y(x) = f(x) \qquad \text{with} \qquad \mathcal{L} = \alpha(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \beta(x)\frac{\mathrm{d}}{\mathrm{d}x} + \gamma(x)$$

for a function $y(x)$ on the interval $a \leqslant x \leqslant b$ with boundary conditions $y(a) = y(b) = 0$. Here, on the interval $a \leqslant x \leqslant b$, the functions α, β and γ are continuous and bounded, α is non-zero and f is bounded.

(a) Describe how the Green's function $G(x;\xi)$ is defined for this differential equation and how it can be used to construct a solution $y(x)$ of $\mathcal{L}y(x) = f(x)$. [3]

(b) Show that the Green's function satisfies the continuity and jump conditions, i.e. that for any ξ with $a < \xi < b$,

(i) $G(x;\xi)$ is continuous at $x = \xi$,

(ii) $$\lim_{x\to\xi^+} \frac{\mathrm{d}G}{\mathrm{d}x} - \lim_{x\to\xi^-} \frac{\mathrm{d}G}{\mathrm{d}x} = \frac{1}{\alpha(x)}\,,$$

where $x \to \xi^+$ ($x \to \xi^-$) denotes the limit towards ξ from above (below). [5]

(c) Use the continuity and jump conditions to construct the Green's function for the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = f(x)\,, \qquad 0 \leqslant x \leqslant \tfrac{\pi}{2}\,, \qquad y(0) = y(\tfrac{\pi}{2}) = 0\,. \qquad (\dagger)$$ [7]

(d) Use this Green's function to construct the solution of Eq. ($\dagger$) for $f(x) = e^x$. [5]

4B

The Fourier transform and its inverse for a function $f(x)$ are given by

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,\mathrm{d}x\,, \qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)e^{ikx}\,\mathrm{d}k\,.$$

(a) Show that an absolutely integrable function $f(x)$ and its Fourier transform $\tilde{f}(k)$ obey Parseval's theorem

$$\int_{-\infty}^{\infty} |f(x)|^2 \mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\tilde{f}(k)|^2 \mathrm{d}k\,.$$ [5]

(b) Compute the Fourier transform of $f(x) = 2\cos^2(ax)$, where $a > 0$ is a real constant.
$\left[\textit{You may use that } 2\pi\delta(k) = \int_{-\infty}^{\infty} e^{-ikx}\mathrm{d}x.\right]$ [6]

(c) Let $a > 0$ be a real constant and

$$f(x) = \begin{cases} \cos x & \text{for } -a \leqslant x \leqslant a \\ 0 & \text{elsewhere} \end{cases} .$$

Show that the Fourier transform of $f(x)$ is given by

$$\tilde{f}(k) = \frac{\lambda k \cos a\ \sin(ak) + \mu \sin a\ \cos(ak)}{k^2 - 1}\,,$$

where λ and μ are numerical constants you should compute. Determine how the behaviour of $\tilde{f}(k)$ in the limit $|k| \to \infty$ depends on the value of a and briefly interpret this dependence. [9]

5A

(a) The matrix

$$\mathbf{R_z}(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad ,$$

rotates a vector about the z-axis by an angle ϕ. Show that, when acting on a vector $\mathbf{v}$ as $\mathbf{v}' = \mathbf{R_z}(\phi)\mathbf{v}$, this matrix preserves the length $|\mathbf{v}'| = |\mathbf{v}|$. [2]

(b) Find the eigenvalues of $\mathbf{R_z}(\phi)$ and determine the values of ϕ for which the eigenvalues are real. Interpret this result geometrically. [5]

(c) Find a matrix $\mathbf{R_y}(\theta)$ which rotates a vector about the y-axis by an angle θ. [1]

(d) A new rotation matrix $\mathbf{R_1}(\theta,\phi) = \mathbf{R_z}(\phi)\mathbf{R_y}(\theta)$ is formed from the product of $\mathbf{R_z}(\phi)$ and $\mathbf{R_y}(\theta)$. Determine if $\mathbf{R_1}(\theta,\phi)$ is

(i) orthogonal,

(ii) unitary,

(iii) Hermitian. [3]

(e) Is the matrix

$$\mathbf{M} = \mathbf{R_1}(\theta,\phi) - \mathbf{R_1}^{\mathrm{T}}(\theta,\phi) \quad ,$$

Hermitian, anti-Hermitian, or neither? [2]

(f) Find the values of θ and ϕ in the ranges $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi \leqslant 2\pi$ for which $\mathbf{R_1}(\theta,\phi)$ maps the vector $\mathbf{v} = (0,0,1)$ to $\mathbf{v}' = (0,1,0)$. For these values of θ and ϕ does the matrix $\mathbf{R_2}(\theta,\phi) = \mathbf{R_y}(\theta)\mathbf{R_z}(\phi)$ also map $\mathbf{v}$ to $\mathbf{v}'$? [4]

(g) Find the general form of a new matrix $\widetilde{\mathbf{R}}(\theta,\phi) = \mathbf{D}'\mathbf{R_1}(\theta,\phi)\mathbf{D}$ which maps the vector $\mathbf{v} = (0,0,c)$ to $\mathbf{v}' = (0,b,0)$, where $\mathbf{D}$ and $\mathbf{D}'$ are diagonal matrices and the values of θ and ϕ are the same as in part (f). [3]

6A

(a) Prove that the eigenvalues of an Hermitian matrix are real. Prove also that the eigenvectors corresponding to distinct eigenvalues of an Hermitian matrix are orthogonal. [7]

(b) Define a unitary matrix. Show that if μ is an eigenvalue of a unitary matrix then $|\mu| = 1$. [2]

(c) Let $\mathbf{I}$ denote the identity matrix and $\mathbf{A}$ be an $n \times n$ complex Hermitian matrix, with a set of n linearly independent eigenvectors $\mathbf{e}_j$ so that

$$\mathbf{A}\,\mathbf{e}_j = \lambda_j\,\mathbf{e}_j\,.$$

For real and positive a, show that $(\mathbf{A} + a\,i\,\mathbf{I})$ is invertible, and that

$$\mathbf{V} = (\mathbf{A} - i\,a\,\mathbf{I}\,)(\mathbf{A} + i\,a\,\mathbf{I}\,)^{-1}$$

is unitary. What are the eigenvalues and eigenvectors of $\mathbf{V}$? [6]

(d) Let $\mathbf{U}$ be a unitary matrix, and assume that 1 is not an eigenvalue, so that $\mathbf{I} - \mathbf{U}$ is invertible. Show that the matrix

$$\mathbf{B} = i(\mathbf{I} + \mathbf{U})(\mathbf{I} - \mathbf{U})^{-1}$$

is Hermitian. [5]

7C

(a) An analytic function of the complex number $z = re^{i\theta}$ can be written as

$$f(z) = u(r,\theta) + iv(r,\theta) \quad ,$$

where u and v are real functions. Show that the Cauchy-Riemann equations for $u(r,\theta)$ and $v(r,\theta)$ in polar coordinates with $r > 0$ are

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r} \quad , \quad \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad .$$ [4]

(b) For $u(r,\theta)$ of the form

$$u(r,\theta) = r^n \cos(n\theta) \ ,$$

where n is a positive integer, use the Cauchy-Riemann equations from part (a) and the boundary condition $f(0) = 0$ to determine the form of $v(r,\theta)$, and hence $f(z)$. [4]

(c) Find and classify the zeroes of

$$f(z) = \cos^3(z) \quad .$$

Does this function have a pole or essential singularity as $|z| \to \infty$? [4]

(d) For each of the following functions, find the series expansion about $z = 0$, $g(z) = \sum_n a_n z^n$, giving an expression for a_n, and determine the radius of convergence.

(i)

$$g(z) = \frac{1}{z_0 + z} \quad ,$$

where z_0 is a complex constant,

(ii)

$$g(z) = \frac{\sin(z)}{1 - z^2} \quad .$$ [8]

8B

Consider the differential equation

$$4x^2(1+x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4x\frac{\mathrm{d}y}{\mathrm{d}x} + (x+3)y = 0\,, \qquad (\dagger)$$

where $y(x)$ is a real function of a real variable x.

(a) Identify all singular points of this differential equation and determine whether they are regular. [2]

(b) Consider a series solution of equation (†) of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+\sigma}\,, \qquad (\ddagger)$$

where $a_0 \neq 0$. Derive the recurrence relation for the coefficients a_k by inserting the series expansion into equation (†). [4]

(c) Determine the two candidate values of σ for which a series solution of the form (‡) may exist. [3]

(d) Using the result of part (b), determine the recurrence relation for the larger of the two candidate values of σ, making sure that you simplify the result as much as possible. Write down the resulting series solution. How many free parameters does this solution have? [4]

(e) Express the series solution from part (d) in closed form. *[Hint: You may find it helpful to Taylor expand* $\ln(1+x)$*.]* [4]

(f) Show that for the smaller of the candidate values of σ, the series (‡) can be terminated such that $a_k = 0$ for $k \geqslant 1$. Write down the solution $y(x)$ of equation (†) corresponding to this case. [3]

9C

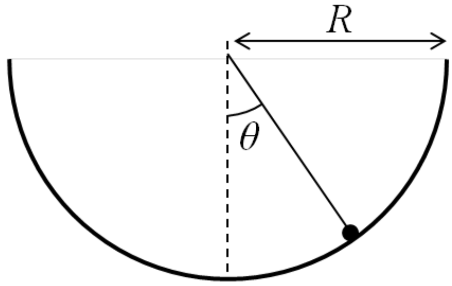
(a) State the *Euler-Lagrange equation* for extremizing the functional

$$F[y] = \int_{\alpha}^{\beta} f(y, y'; x)\, \mathrm{d}x\,,$$

where $y = y(x)$ and $y' = \mathrm{d}y/\mathrm{d}x$. [1]

(b) If $f(y, y')$ has no explicit dependence on x, find a first-integral of the problem, i.e. a nonzero quantity h constructed from f for which $\mathrm{d}h/\mathrm{d}x = 0$. [3]

(c) Consider a circular cylinder of radius R with its axis parallel to the ground. Now consider a ramp which is the inner surface of the half-cylinder formed when this cylinder is cut in a horizontal plane along its axis and the top half is removed. Work in cylindrical polar coordinates (r, θ, z) with $\theta = 0$ pointing vertically downwards, as in the figure below, and the z-axis along the axis of the cylinder.



(i) Assuming that $z(\theta)$ is a single-valued function of θ, show that the length of a path $z(\theta)$ along the ramp is given by

$$L[z] = \int_{\theta_1}^{\theta_2} \sqrt{R^2 + z'^2}\, \mathrm{d}\theta\,.$$

Find $z(\theta)$ for the shortest path between a point on the edge of the ramp at $(z, \theta) = (z_1, -\pi/2)$ and a point on the opposite edge of the ramp at $(z_2, \pi/2)$ with $z_2 > z_1$. [8]

(ii) Now suppose that a skater initially at rest at $(z, \theta) = (z_1, -\pi/2)$ wants to reach $(z_2, \pi/2)$ in the shortest possible time. Assuming conservation of energy (neglecting any resistive forces), show that the time taken for a path $z(\theta)$ along the ramp is

$$T[z] = \frac{1}{\sqrt{2gR}} \int_{\theta_1}^{\theta_2} \sqrt{\frac{R^2 + z'^2}{\cos\theta}}\, \mathrm{d}\theta\,,$$

where g is the acceleration due to gravity. Find an expression for $\mathrm{d}z/\mathrm{d}\theta$ for the path which minimizes T. [You do *not* need to solve this equation for $z(\theta)$.] In which direction should the skater head initially? [8]

10C (a) Consider a general second-order differential operator of the form

$$\tilde{\mathcal{L}} = -\frac{\mathrm{d}}{\mathrm{d}x}\left(\tilde{a}(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) - \tilde{b}(x)\frac{\mathrm{d}}{\mathrm{d}x} - \tilde{c}(x)\,, \qquad (\dagger)$$

where $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ are real functions and $\tilde{a}(x) > 0$ for $\alpha \leqslant x \leqslant \beta$. Explain how this can be related to a Sturm-Liouville operator

$$\mathcal{L} = w(x)\tilde{\mathcal{L}} = -\frac{\mathrm{d}}{\mathrm{d}x}\left(a(x)w(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) - w(x)c(x)\,, \qquad (\ddagger)$$

where $a(x)w(x) > 0$ for $\alpha < x < \beta$, specifying how $a(x), w(x), c(x)$ are related to $\tilde{\mathcal{L}}$. [4]

(b) Consider a Sturm-Liouville operator $\mathcal{L}$ of the form ($\ddagger$), where the weight function $w(x) > 0$ for x in the range $\alpha \leqslant x \leqslant \beta$. The eigenvalue equation for real functions $y(x)$ defined for $\alpha \leqslant x \leqslant \beta$ is

$$\mathcal{L}y(x) = \lambda w(x)y(x)\,,$$

with boundary conditions $y(\alpha)\dot{y}(\alpha) = y(\beta)\dot{y}(\beta) = 0$, where $\dot{y} = \frac{\mathrm{d}y}{\mathrm{d}x}$. Show that the lowest eigenvalue $\lambda_{\min}$ can be obtained by minimizing a functional of the form

$$\Lambda[y] = \frac{F[y]}{G[y]}$$

over functions y satisfying the relevant boundary conditions, where F and G are functionals that depend linearly on $\mathcal{L}$ and w respectively. Specify the precise form of F and G. [4]

(c) Consider functions $y(x)$ defined for $0 \leqslant x \leqslant 1$ that satisfy

$$(1+x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (3+2x)\frac{\mathrm{d}y}{\mathrm{d}x} + \lambda y = 0\,,$$

with $y(0) = 0$ and $\dot{y}(1) = 0$. By applying the results of part (a), rewrite this as a Sturm-Liouville eigenvalue equation. Hence obtain an upper bound on the lowest eigenvalue λ for such functions by considering a trial function of the form $x\exp(-x)$. [9]

(d) Explain how you could attempt to improve this bound by considering functions of the following forms:

(i) $x\exp(-x) + Cx^n\exp(-Ax)$ for positive integers n and a suitable choice of A which you should specify.

(ii) $x\exp(-x) + C\sin(Bx)$ for suitable values of B which you should specify.

[*You are not required to obtain any improved bounds.*] [3]

END OF PAPER