UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Wednesday, 16 June, 2021 11:30 am to 14:30 pm

MATHEMATICS (2)

Before you begin read these instructions carefully:

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks. The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

*You may upload your answers to **all** of section A, and to no more than **five** questions from section B. Answers must be **handwritten with a dark pen on paper**, so as to scan clearly, and then scanned into PDF files as follows. Your section A solutions should be scanned and uploaded together into a **single** PDF file named A. For **each** of your five section B solutions you should upload one PDF file named after the question, (for example use filename 13S for your response to question 13S). So for the whole paper you may upload up to **six** PDF files.*

*You should upload your files within 40 minutes after the end of the three hour exam. **Please ensure your scans are legible.** If the process of scanning and uploading takes you longer than 40 minutes you should email your college tutor to explain why.*

Please click this link https://www.maths.cam.ac.uk/NSTIA_P2_Coversheet and fill out the form to indicate which questions you have attempted and uploaded.

Calculators are not permitted in this examination. This is a closed-book examination, with no outside sources permitted, which must be done under self-imposed examination conditions.

SECTION A

1 Find the eigenvalues and eigenvectors of the 2×2 matrix

$$\begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix},$$

where u is real. [2]

2 Find the vector area of the rectangle $ABCDA$ whose corners A, B, C and D are, respectively, at the four points [2]

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

3 Find the general solution of [2]

$$\frac{dy}{dx} + y = e^{2x}.$$

4 Given that $\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$ calculate $\int_{-\infty}^{+\infty} x^n e^{-ax^2} dx$ for any $a > 0$ and n equal to 0 and 1. [2]

5 Let $L(x, v) = \frac{1}{2}mv^2 - V(x)$, where $m > 0$ and V is a given function of x. Find v which maximizes the value of $H = pv - L(x, v)$, as v varies with x and p fixed. [1]

Give this maximum value. [1]

6 Let

$$\boldsymbol{F} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad \boldsymbol{G} = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}.$$

Calculate $\boldsymbol{F} \times \boldsymbol{G}$ and $\nabla \cdot \left(\boldsymbol{F} \times \boldsymbol{G}\right)$. [2]

7 A random variable X can take on only the values $\{0, 1, 2, \dots\}$ with probabilities

$$P(X = n) = ce^{-n}$$

where c is a real constant. Find c, and then find, for each positive integer N, [2]

$$P(X \geqslant N)\,.$$

8 Without differentiating, find the n-th term in the Taylor expansion about $x = 0$ of [2]

$$f(x) = \frac{1}{1+x^2}\,.$$

9 For which real number a does $u(x,t) = f(x + at)$ solve the equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0\,,$$

for arbitrary differentiable f. [2]

10 Let

$$\boldsymbol{F} = \begin{pmatrix} e^y \cos x \\ e^x \cos y \\ e^z/(1+z^2) \end{pmatrix}$$

and let D be the unit disc centred at the origin in the $x - y$ plane, with normal vector

$$\mathbf{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\,.$$

Calculate the surface integral $\int_D \boldsymbol{F} \cdot \mathbf{n}\, dS$. [2]

SECTION B

11W

Two lines are defined by the following equations for position vectors: $\boldsymbol{r}_1(\lambda) = \mathbf{a}_1 + \lambda\mathbf{n}_1$ and $\boldsymbol{r}_2(t) = \mathbf{a}_2 + t\mathbf{n}_2$, where $\mathbf{a}_1 = (1,0,0)$, $\mathbf{a}_2 = (0,0,1)$, $\mathbf{n}_1 = (1,1,0)$, $\mathbf{n}_2 = (1,1,1)$ and λ and t are real parameters.

(a) Find the distance between the two points given by the position vector $\boldsymbol{r}_1$ evaluated at $\lambda = 1$ and by the position vector $\boldsymbol{r}_2$ evaluated at $t = -1$. [2]

(b) Find the shortest distance, s, between the two lines. [3]

(c) By using the scalar product (and without use of the vector product):

(i) derive, in terms of $\mathbf{a}_2$ and $\mathbf{n}_2$, the expression for the shortest distance, d_0, between the origin $(0,0,0)$ and the second line $\boldsymbol{r}_2(t)$; [4]

(ii) calculate the numerical value of d_0. [1]

(d) (i) By using the vector product, find the shortest distance, $d(t)$, between the first line and an arbitrary point on the second line. [4]

(ii) Sketch the graph of $d(t)$ and find equations for asymptotes if any. [3]

(iii) Find the value of $t = t_{\text{min}}$ for which $d(t)$ achieves its minimum, find $d_{\text{min}} = d(t_{\text{min}})$ and compare your answer with that in (b). [3]

12X

Consider a scalar field in the $x - y$ plane defined by a function $f(x,y)$ where f, x, and y are all real.

(a) Explain what is meant by a *stationary point* and sketch the contours of f in a neighbourhood of a maximum, a minimum, and a saddle point. Explain why $\nabla f = \mathbf{0}$ at a stationary point. [4]

(b) Find the locations of, and the values of f at, the stationary points of

$$f(x,y) = 2\frac{y - x^2}{1 + x^2 + y^2}\,.$$

[5]

(c) Sketch the $x - y$ plane to show the contours of constant f for $f = 0$, $f = +1$ and $f = -1$. Hence or otherwise determine the nature of each stationary point of f. [4]

(d) Complete your sketch to show contours of constant f. [3]

(e) A particle starts from $(x,y) = (1,1)$ and travels along a trajectory such that the direction of motion is parallel to ∇f. Add this trajectory to your sketch. (*Derivation of the formula for the trajectory is not required.*) [4]

13S In this question, consider the vector field given by

$$\mathbf{F}(\mathbf{r}) = \begin{pmatrix} -ay + e^{-r^2}x \\ ax + e^{-r^2}y \\ e^{-r^2}z \end{pmatrix}$$

where $r^2 = x^2 + y^2 + z^2$ and a is real.

(a) Calculate $\nabla \times \mathbf{F}$. [2]

(b) Find $\nabla \cdot \nabla \times \mathbf{F}$. [1]

(c) For which values of a is $\nabla \times \mathbf{F} = 0$? [1]

(d) Calculate $\int_\Gamma \mathbf{F} \cdot d\mathbf{r}$ along the path

$$\Gamma: \quad \mathbf{r}(t) = \begin{pmatrix} \cos(2\pi t^n) \\ \sin(2\pi t^n) \\ 1 \end{pmatrix} \qquad 0 \leqslant t \leqslant 1\,,$$

for all $n = 1, 2, 3 \ldots$. [5]

(e) Consider the surface integral $I = \int_D \nabla \times \mathbf{F} \cdot d\mathbf{S}$ where D is the disc $x^2 + y^2 \leqslant 1, z = 1$ and $d\mathbf{S}$ is oriented along $\mathbf{k}$, the unit vector in the z direction. Calculate $\nabla \times \mathbf{F} \cdot \mathbf{k}$, and hence find I. [5]

(f) For all real values of the parameter a, either find a scalar field $\Phi(\mathbf{r})$ such that $\mathbf{F} = \nabla\Phi$ or show that no such field exists. [6]

14V

A coffee machine is supposed to serve drinks containing 300 ml of liquid. Coffee machines are verified by a test consisting of a single trial (producing a single cup of coffee), and the machine passes if it serves between 290 and 310 ml of liquid. A particular coffee machine makes random and independent servings in the range between 270 and 330 ml. The volume of liquid in each serving can be considered as a continuous random variable, v.

(a) Suppose the probability density function for v is

$$P(v) = \frac{\pi}{120} \cos\left(\frac{\pi(v - 300\,ml)}{60\,ml}\right) (ml)^{-1}$$

between 270 ml and 330 ml (zero otherwise).

(i) Sketch the graph of the probability density function for v. [3]

(ii) Determine the probability that the machine passes the verification test. [2]

(iii) Prove that if $0 < r < 1$ then $\displaystyle\sum_{j=1}^{\infty} jr^{j-1} = (1-r)^{-2}$. [3]

(iv) If the verification test is repeated, use the formula in (iii) to calculate the expected number of tests in order for the fault to be discovered. [4]

(b) Two coffees are made with the machine. Calculate the probability that the total volume of liquid is at least 630 ml. [8]

15Y

(a) Solve the equation

$$x^2 \frac{dy}{dx} + xy = x^2 + y^2, \quad x > 0,$$

subject to the boundary condition $y = 0$ at $x = e$. [6]

(b) Consider two simultaneous first-order linear differential equations

$$\begin{cases} \frac{du}{dt} & = 4v + 2\,, \\ \frac{dv}{dt} & = u\,. \end{cases}$$

(i) Find a second-order differential equation for v. [1]

(ii) Find the general solution for v. [3]

(iii) Find the general solution for u. [2]

(c) (i) Find the general solution of [5]

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 3x\,. \qquad (\dagger\dagger)$$

(ii) Find the solution of equation ($\dagger\dagger$) subject to the initial conditions: $y(0) = 1$ and $y'(0) = 0$. [3]

16R

(a) Let

$$\psi(x, y, z) = \frac{a}{2}\ln(x^2 + y^2)\,,$$

where a is a positive parameter.

(i) Find the vector field $\mathbf{u} = \nabla \times (\psi \hat{\mathbf{e}}_\mathbf{z})$, where $\hat{\mathbf{e}}_\mathbf{z}$ is the unit vector in the z direction. [3]

(ii) Calculate $\nabla \cdot \mathbf{u}$. [3]

(iii) Express the function ψ in terms of cylindrical polar coordinates (r, θ, z) and calculate the vector field $\mathbf{u} = (u_r, u_\theta, u_z) = \nabla \times (\psi \hat{\mathbf{e}}_\mathbf{z})$ in cylindrical polar coordinates by using the definition

$$\nabla \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{e}}_\mathbf{r} & r\hat{\mathbf{e}}_\theta & \hat{\mathbf{e}}_\mathbf{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}\,,$$

where $\mathbf{A} = (A_r, A_\theta, A_z)$ is an arbitrary vector field and $\hat{\mathbf{e}}_\mathbf{r}$, $\hat{\mathbf{e}}_\theta$ and $\hat{\mathbf{e}}_\mathbf{z}$ are the unit vectors in the corresponding directions. [4]

(b) For the vector field $\mathbf{F} = (0, (y+3)^2, z^2)$ calculate the integral of the divergence, $\nabla \cdot \mathbf{F}$, over the volume of the triangular pyramid bounded by the four planes: $x = 0$, $y = 0$, $z = 0$ and $x + y + z/2 = 1$. [10]

17Z

(a) Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be square matrices. Define $\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{AB})^{\mathrm{T}}$.

(i) By any method, demonstrate that if $\boldsymbol{A}$ and $\boldsymbol{B}$ are orthogonal then $\boldsymbol{C}$ is also orthogonal. [2]

(ii) By using the suffix notation, demonstrate that if $\boldsymbol{B}$ is anti-symmetric then $\boldsymbol{C}$ is also anti-symmetric. [3]

(b) Consider the following set of simultaneous equations for x, y, z:

$$\begin{cases} x + \lambda y - 1 &= 0 \\ \lambda x - z - 2 &= 0 \\ 4y + z - 1 &= 0\,. \end{cases} \qquad (\dagger)$$

(i) Rewrite ($\dagger$) in a matrix form, $\boldsymbol{Lr} = \boldsymbol{b}$, and give explicit expressions for $\boldsymbol{L}, \boldsymbol{r}$ and $\boldsymbol{b}$. [2]

(ii) Determine the values of the real parameter λ for which the set of equations ($\dagger$) has solutions. [4]

(iii) For these values of λ find all solutions for $\boldsymbol{r}$. [4]

(c) Given the matrix $\boldsymbol{X} = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$, find all real matrices $\boldsymbol{Y}$ for which

(i) $\boldsymbol{XY} = \boldsymbol{YX}$, [3]

(ii) $\boldsymbol{XY} = \boldsymbol{YX}$, $\det\boldsymbol{X} = \det\boldsymbol{Y}$ and $\mathrm{Tr}\,\boldsymbol{X} = \mathrm{Tr}\,\boldsymbol{Y}$. [2]

18R Consider the function

$$F(a_0, a_1, \dots a_N) = \int_{-L}^{L} \left[f(x) - \frac{a_0}{2} - \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi x}{L}\right) \right]^2 dx\,,$$

where f is an *even* function of x of period $2L$.

(a) Obtain the values of $\{a_n\}$, $n = 0, 1, \dots N$, which minimize the function $F(a_0, a_1 \dots a_n)$ and state the connection with Fourier series. [6]

(b) For the case

$$f(x) = \begin{cases} x\,, & 0 \leqslant x \leqslant L \\ -x\,, & -L < x < 0 \end{cases}$$

calculate the Fourier series of f. [2]

(c) Deduce that [5]

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots .$$

(d) State Parseval's theorem. [3]

(e) Find the value of [4]

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)^4}\,.$$

19T*

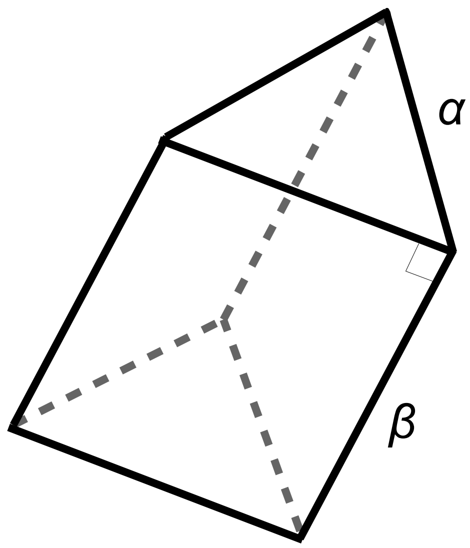
(a) The following function of 2 variables has a stationary point at the origin (0,0):

$$f(x,y) = 1 + 2x^2 + 3xy + y^2 + xy^2 + 5y^3\,.$$

(i) Write down the Hessian matrix for this stationary point. [3]

(ii) Determine if this stationary point is a local maximum, a local minimum, or a saddle point. Justify your answer. [3]

(b) A builder is constructing a perpendicular triangular prism, consisting of 2 equilateral triangles with sides of length α, whose corners are connected by line segments of length β that are perpendicular to the triangular faces (as shown in the diagram). Thus the prism has 5 faces (2 triangles and 3 rectangles).



Use the method of Lagrange multipliers to determine the values of α and β which maximize the volume V, for fixed value of the total external area of the prism $A = 4.5\,\mathrm{m}^2$. [14]

[*You need not prove that the extremal value you find is a maximum.*]

20W*

The temperature $\Theta(x,t)$ of a finite bar $0 < x < L$ obeys the heat conduction equation,

$$\frac{\partial^2\Theta}{\partial x^2} = \frac{1}{\kappa}\frac{\partial\Theta}{\partial t}\,.$$

The initial temperature profile is given by $\Theta(x,0) = T_0\sin^2(\pi x/L)$ where the parameters κ and T_0 are positive. Find $\Theta(x,t)$ for $t > 0$ subject to the given boundary conditions in the following two cases:

(a) $\frac{\partial\Theta}{\partial x}(0,t) = \frac{\partial\Theta}{\partial x}(L,t) = 0$; [8]

(b) $\Theta(0,t) = \Theta(L,t) = 0$. [12]

END OF PAPER