UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Wednesday, 9 June, 2021 11:30 am to 14:30 pm

MATHEMATICS (1)

Before you begin read these instructions carefully:

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks. The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

You may upload your answers to ***all*** *of section A, and to no more than* ***five*** *questions from section B. Answers must be* ***handwritten with a dark pen on paper****, so as to scan clearly, and then scanned into PDF files as follows. Your section A solutions should be scanned and uploaded together into a* ***single*** *PDF file named A. For* ***each*** *of your five section B solutions you should upload one PDF file named after the question, (for example use filename 13Y for your response to question 13Y). So for the whole paper you may upload up to* ***six*** *PDF files.*

You should upload your files within 40 minutes after the end of the three hour exam. ***Please ensure your scans are legible.*** *If the process of scanning and uploading takes you longer than 40 minutes you should email your college tutor to explain why.*

Please click this link https://www.maths.cam.ac.uk/NSTIA_P1_Coversheet *and fill out the form to indicate which questions you have attempted and uploaded.*

Calculators are not permitted in this examination. This is a closed-book examination, with no outside sources permitted, which must be done under self-imposed examination conditions.

SECTION A

1 If $y(x) = \sin x^3 + \cos 2x$, find $\frac{d}{dx}(y^2)$. [2]

2 Factorize $x^2 - x - 2$ and hence find all x for which $x^2 - x - 2 < 0$. [2]

3 Find all θ in the interval $0 \leqslant \theta \leqslant \pi$ such that [2]

$$\sin^4\theta + \cos^4\theta = \frac{1}{2}\,.$$

4 Calculate the values of $2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ and $2 - 1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \ldots$. [2]

5 Find the minimum and maximum values of xe^{-2x^2} for $0 \leqslant x \leqslant 1$. [2]

6 Find the cosine and sine of the angle θ between the line segment from the point $(1, 1, 1)$ to the origin and the line segment from the point $(1, 1, 1)$ to the point $(2, 2, 1)$. [2]

7 How many real roots does the polynomial $x^3 - 5x - 1$ have? [2]

8 Find the indefinite integral of $x^3e^{-x^4}$. [2]

9 Given $t = \tan\frac{\theta}{2}$, give expressions for $\tan\theta$ and $\sec^2\theta$ in terms of t. [2]

10 In the $x - y$ plane, find the equation for the tangent line to the ellipse $9x^2 + 16y^2 = 25$ at the point $(1, 1)$. [2]

SECTION B

11W

(a) Find the real and imaginary parts of the following complex numbers:

(i) $(1/2 + 2i)^2$, [2]

(ii) $(1/2 + 2i)^4$. [2]

In the remainder of the question, a point $z = t + i/t$ moves in the complex plane as the real parameter t increases continuously from $t = 1/2$ to $t = 2$.

(b) In the following three cases, sketch the trajectories of the given points as t varies, showing the direction of their motion by arrows, and giving the coordinates of the initial and final positions, and also of points (if any) where the trajectories cross the axes:

(i) z and z^*, where z^* is the complex conjugate of z ; [2]

(ii) $u = z^2$; [4]

(iii) $v = z^4$. [8]

(c) For $v = z^4$, find a relation between the real and imaginary parts of v, and thus demonstrate that $\mathrm{Re}(v)$ is a quadratic function of $\mathrm{Im}(v)$. [2]

12R

(a) An isotropic spherical body of radius $R > 0$ is centred at the origin. Its mass density, $\rho(r)$, depends on the distance, $r > 0$, from the origin according to the following expression,

$$\rho(r) = Ar^{-2}\left(1 - e^{-r/r_0}\right),$$

where A and r_0 are positive parameters.

(i) Write down the volume element dV in spherical polar coordinates (r, θ, φ). By integrating $\rho(r)$ over the volume of the body, find the mass, M, of the body and thus demonstrate that

$$M = 4\pi ARf(r_0/R),$$

where you should determine the function $f(r_0/R)$. [3]

(ii) Find approximate expressions for $f(r_0/R)$ in two limiting cases: $r_0/R \to \infty$ and $r_0/R \to 0$. [4]

(iii) Sketch the graph of $f(r_0/R)$. You may assume that $f(r_0/R)$ does not have stationary points for $r_0/R > 0$. [3]

(b) The coordinates (x, y, z) of the points inside a closed shape of volume V obey the following inequalities:

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} < \frac{z}{d} + 1 \quad \text{and} \quad z < 0,$$

where a and d are positive parameters. Write down the volume element dV in cylindrical polar coordinates (r, θ, z). Express the volume V as a triple integral and calculate this integral. [10]

13Y

(a) Consider the equation

$$\frac{d^2y}{dx^2} + y = \frac{1}{2}\left[1 - \cos(2x)\right]. \qquad (\dagger)$$

(i) Find the general solution to the corresponding homogeneous equation; [3]

(ii) Find a particular solution of equation (†); [3]

(iii) Write down the solution of equation (†) subject to the boundary conditions: [5]

$$y(0) = 0 \qquad \text{and} \qquad y(\pi/2) = 0.$$

(b) Find explicitly all solutions of the differential equation

$$y' = 2y + 4y^5.$$

[9]

14Q

(a) The magnitude of the gravitational force between two point masses m_1, m_2, which are separated by a distance $r > 0$ in three dimensional space, is

$$F(r, m_1, m_2) = G\frac{m_1 m_2}{r^2}\,,$$

where G is a positive constant.

(i) Find the differential expression for the small change in force due to small changes in the masses of the particles and the distance. [3]

(ii) What is the fractional change in distance if there is no change in force when the masses of both particles increase by 1%? [4]

(b) For $\mathbf{r} = (x, y, z)$, let $f(\mathbf{r}) = c/r^2$ for constant c. What is the rate of change of f in the direction $(1, 1, 0)$ at the point $(1, 1, 1)$? [5]

(c) Calculate the determinant

$$J = \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{vmatrix}$$

for $x = r\cos\theta$ and $y = r\sin\theta$. [3]

(d) Show that the function

$$\phi(x,t) = \frac{1}{\sqrt{4\pi\sigma^2 t}}\exp\left[-\frac{(x-x_0)^2}{4\sigma^2 t}\right]$$

satisfies

$$\frac{\partial \phi}{\partial t} = \sigma^2\frac{\partial^2\phi}{\partial x^2}$$

for $t > 0$, x_0 and σ real and $\sigma^2 \neq 0$. [5]

15V

(a) State Taylor's theorem by giving the series expansion about $x = a$ of a function $f(x)$ that is n times differentiable, showing the first n terms, together with an expression for the remainder term R_n. [4]

(b) Show that the Taylor series expansion of $f(x) = \frac{\cosh x}{\cos x}$ about the point $x = 0$ as far as the term in x^4 is $f(x) \approx 1 + x^2 + \frac{x^4}{2}$. [4]

(c) Find the Taylor series expansion of $g(x) = \cosh(\ln x)$ about the point $x = 2$ as far as the term in $(x-2)^3$. [6]

(d) Show that the Taylor series expansion of $h(x) = \ln(2 - e^x)$ about the point $x = 0$ as far as the term in x^3 is $h(x) \approx -x - x^2 - x^3$ and state the range of x for which the infinite series converges. [6]

16T

Three standard 6-sided dice are tossed once onto a table.

(a) What are the mean and variance of the sum of the values showing on the dice? [5]

[Hint: start by calculating the mean and variance of a single die. How is this related to the total?]

(b) Calculate the probability of the following events:

(i) S: "All three dice show the same numerical value." [2]

(ii) D: "All three dice show different numerical values." [2]

(iii) T: "Two of the dice are the same and one is different." [3]

(iv) F: "The sum of the dice is five or less." [3]

(c) Calculate the conditional probability $P(F|T)$ using Bayes' theorem. [5]

17W

Let $F(x)$ be the real-valued function defined as

$$F(x) = \int_a^x f(u)\,du\,,$$

where a is a real constant and $f(u)$ is a continuous and real-valued function of u.

(a) Write down an expression for dF/dx in terms of f. [1]

(b) For $f(u) = 1/u$:

(i) find $F(x)$ for $x < 0$ if $a = -1$; [2]

(ii) find a function $G(x)$ such that $dG/dx = 1/x$ for all x (excluding $x = 0$), and sketch the graph of this function. [3]

(c) For $f(u) = 1/\sqrt{u^2 - 1}$:

(i) by using the substitution $z = u + \sqrt{u^2 - 1}$ or otherwise, find $F(x)$ for $x > 1$ if $a = 1$; [3]

(ii) find a function $R(x)$ such that $dR/dx = 1/\sqrt{x^2 - 1}$ for all x (excluding $-1 \leqslant x \leqslant 1$), and sketch the graph of this function. [6]

(d) Find $F(x)$ for the case

$$f(u) = \frac{\sin(2u)}{\sin^2 u + \ln(u)} + \frac{1}{u(\sin^2 u + \ln(u))}\,,$$

with $a = \pi$ and $x > 1$. [5]

18Z

(a) Given the matrix

$$\boldsymbol{M} = \begin{pmatrix} a^2 & b & -3 \\ -b & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix},$$

determine all values of the real parameters a and b for which

(i) $\mathrm{Tr}\,(\boldsymbol{M}) = 3$; [1]

(ii) $\mathrm{Tr}\,(\boldsymbol{M}) = 3$ and $\boldsymbol{M}$ is invertible. [2]

(b) Given the matrix

$$\boldsymbol{A} = \begin{pmatrix} -5 & -4 \\ 4 & 5 \end{pmatrix},$$

(i) find a matrix $\boldsymbol{B}$ such that $\boldsymbol{AB} = \boldsymbol{BC}$ where $\boldsymbol{C}$ is a square diagonal matrix; [5]

(ii) find $\boldsymbol{B}^{-1}$. [3]

(c) Define a matrix $\boldsymbol{Z}$ by $\boldsymbol{Z} = \boldsymbol{YXY}^{-1}$, where $\boldsymbol{Y}$ and $\boldsymbol{X}$ are square matrices and $\boldsymbol{X}$ is antisymmetric. Demonstrate that $\mathrm{Tr}\,(\boldsymbol{Z}) = 0$. [2]

(d) Given the matrix

$$\boldsymbol{L} = \begin{pmatrix} -1 & b \\ c & d \end{pmatrix},$$

determine the real parameters b, c and d for which the matrix is

(i) orthogonal; [3]

(ii) orthogonal but is *not* a rotation. [2]

(e) Write the matrix $\begin{pmatrix} a & b \\ c & 3 \end{pmatrix}$ as the sum of a symmetric and an antisymmetric matrix, where a, b and c are real parameters. [2]

19T*

(a) Consider the following function:

$$f(x) = \begin{cases} A|x|^p \sin(\omega x)\,, & x \neq 0, \\ 0\,, & x = 0\,, \end{cases}$$

where the domain x is real, and $A > 0$, $\omega > 0$, and p are real-valued parameters.

(i) For which values of the parameters is the function $f(x)$ continuous at $x = 0$? [6]

(ii) For which values of the parameters is the function $f(x)$ differentiable at $x = 0$? [6]

(iii) Show that for $p < -1$, the following sum is convergent: [4]

$$\sum_{n=1}^{\infty} f(n)\,.$$

[*You may quote and use standard results on convergence/divergence of series.*]

(b) Approximate the following function in the limit $x \gg 1$:

$$g(x) = \frac{1}{x + 3x^3}\,.$$

Give the leading term and the order of the error by using big-O notation. [4]

20Q*

(a) Evaluate the following integral by finding the limiting value of a Riemann sum: [9]

$$J = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sin x \, dx = \mathrm{Im} \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} e^{ix} \, dx\,.$$

(b) Differentiate the function

$$f(x) = \int_{e^{\alpha x}}^{e^{\beta x}} \sin(xt^2)\, dt\,, \qquad \text{where } \alpha, \beta \text{ and } x \text{ are real,}$$

with respect to x. It is not necessary to evaluate the integrals that arise. What is the value of the derivative for $\alpha = \beta$? [5]

(c) Evaluate the integral

$$I = \int_0^{\infty} \frac{e^{-\alpha x^2} - e^{-\beta x^2}}{x}\, dx\,, \qquad \text{where} \quad \alpha > 0\,,\ \beta > 0\,,$$

by considering $\frac{\partial I}{\partial \alpha}$ and $\frac{\partial I}{\partial \beta}$. [6]

END OF PAPER