UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB

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MATHEMATICS FORMATIVE ASSESSMENT

Before you begin read these instructions carefully:

This is a closed-book assessment. You have THREE HOURS to complete the assessment.

You may submit answers to no more then ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, ***6B****).*
Your answer for each question should be submitted as a single .pdf file, labelled with your Blind Grade Number and the question number and letter, following the Moodle instructions.

1B

(a) Let $(\rho,\, \phi,\, z)$, $0 \leqslant \rho < \infty$, $0 \leqslant \phi < 2\pi$, $-\infty < z < \infty$, denote cylindrical coordinates defined in terms of Cartesian coordinates $(x,\, y,\, z)$ by

$$x = \rho\cos\phi\,, \qquad y = \rho\sin\phi\,, \qquad z = z\,.$$

(i) Show that $(\rho\,,\ \phi\,,\ z)$ defines an orthogonal coordinate system. Determine the scale factors h_ρ, h_ϕ, h_z. [5]

(ii) Let $\boldsymbol{e}_\rho$, $\boldsymbol{e}_\phi$ and $\boldsymbol{e}_z$ denote the unit basis vectors associated with the cylindrical coordinates. Calculate their cross products $\boldsymbol{e}_\rho \times \boldsymbol{e}_\phi$, $\boldsymbol{e}_\phi \times \boldsymbol{e}_z$, $\boldsymbol{e}_z \times \boldsymbol{e}_\rho$. Express the results in terms of the basis $\{\boldsymbol{e}_\rho,\ \boldsymbol{e}_\phi,\ \boldsymbol{e}_z\}$ and thus determine the handedness of the basis. [4]

(iii) Calculate the line element $d\boldsymbol{r}$ and its square $|d\boldsymbol{r}|^2$ in cylindrical coordinates. [2]

(iv) Show that the distance between two points $(\rho,\ \phi,\ z)$ and $(\rho',\ \phi',\ z')$ is

$$R = \sqrt{h(\rho, \rho', \phi, \phi') + j(z, z')}\,,$$

where h and j are functions you should determine. [5]

(b) Consider spherical coordinates $(r,\ \theta,\ \phi)$, $0 \leqslant r < \infty$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi < 2\pi$, defined by

$$x = r\sin\theta\,\cos\phi\,, \qquad y = r\sin\theta\,\sin\phi\,, \qquad z = r\cos\theta\,,$$

and let $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$, $\boldsymbol{e}_\phi$ denote the unit basis vectors associated with the spherical coordinates. Show that

$$\frac{\partial}{\partial\theta}\boldsymbol{e}_r = \alpha\boldsymbol{e}_\theta\,,$$

where α is a factor you should determine. [4]

2B

(a) Legendre's differential equation may be written as

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0\,, \qquad (\dagger)$$

where $n \in \mathbb{R}$, x is in the range $-1 \leqslant x \leqslant 1$ and the solutions are normalized such that $y(1) = 1$.

(i) Use the ansatz $y = \displaystyle\sum_{m=0}^{\infty} a_m x^m$ to derive a recursive formula for the coefficients a_m to solve Eq. (†). [4]

(ii) Show that Eq. (†) is solved by a polynomial $P_n(x)$ of order n, when n is a nonnegative integer. Find P_0, P_1 and P_2 explicitly. [3]

(b) Laplace's equation in spherical coordinates for an axisymmetric function $U(r,\theta)$ (i.e. no ϕ dependence) is given by

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right) = 0\,. \qquad (\ddagger)$$

(i) Use separation of variables to determine all solutions to Eq. (‡) of the form $U(r,\theta) = R(r)\,F(\theta)$. State the form of the general solution. [7]

(ii) Find the specific solution $U(r,\theta)$ that satisfies the boundary conditions

$$U(r,\theta) \to v_0 r\cos\theta \qquad \text{as } r \to \infty\,,$$

$$\frac{\partial U}{\partial r} = 0 \qquad \text{at } r = r_0\,,$$

where v_0 and r_0 are constants and $r_0 > 0$. [6]

[Hint: You may assume in part (b) without proof that the polynomials P_n from part (a) are the only bounded solutions of Legendre's equation (†) and that the P_n are orthogonal.]

3C

(a) Three matrices are defined as

$$\boldsymbol{\tau_1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \boldsymbol{\tau_2} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \boldsymbol{\tau_3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad .$$

Calculate the squares $(\boldsymbol{\tau_1})^2$, $(\boldsymbol{\tau_2})^2$ and $(\boldsymbol{\tau_3})^2$. From this, deduce the general form of the powers $(\boldsymbol{\tau_j})^n$ for n a positive integer, distinguishing the even and odd cases. [4]

(b) A matrix exponential is defined as

$$e^{\mathbf{M}} = \sum_{j=0}^{\infty} \frac{(\mathbf{M})^j}{j!} \quad ,$$

where $(\mathbf{M})^0$ is the identity matrix. Using this definition and the previous result calculate the form of $\mathbf{A_1} = e^{i\alpha_1 \boldsymbol{\tau_1}}$, where α_1 is a real number. Similarly, determine the forms of $\mathbf{A_2} = e^{i\alpha_2 \boldsymbol{\tau_2}}$ and $\mathbf{A_3} = e^{i\alpha_3 \boldsymbol{\tau_3}}$.

[*Hint: It will help to recall the Taylor expansions for* $\cos(\theta)$ *and* $\sin(\theta)$.] [5]

(c) Find the determinants of the individual matrices $\mathbf{A_j}$ and hence the determinant of the matrix product $\mathbf{A} = \mathbf{A_1 A_2 A_3}$. [2]

(d) Consider the spatial coordinate system $\mathbf{x} = (x_1, x_2, x_3)$ and the matrix

$$\mathbf{M} = i \sum_{j=1}^{3} x_j \boldsymbol{\tau_j} \quad .$$

Calculate $\mathrm{Det}(\mathbf{M})$ and $\mathrm{Det}(\mathbf{A_1^\dagger M A_1})$. [3]

(e) Provide a geometric interpretation of the action of $\mathbf{A_1}$ on the coordinates $\mathbf{x} = (x_1, x_2, x_3)$ if

$$\mathbf{M'} = \mathbf{A_1^\dagger M A_1} \quad ,$$

where

$$\mathbf{M'} = i \sum_{j=1}^{3} x'_j \boldsymbol{\tau_j} \quad .$$

Also interpret the action of $\mathbf{A_2}$ and $\mathbf{A_3}$ on the coordinates. [6]

4C

(a) Determine whether the point $x = 0$ is an ordinary, singular, or regular singular point of the second-order differential equation

$$x(1-x)\frac{d^2y(x)}{dx^2} + (1-x)\frac{dy(x)}{dx} + y(x) = 0 \quad . \qquad (\dagger)$$

[3]

(b) Find the roots of the indicial equation and write the general form of the solutions to Eq. (†). [4]

(c) Assuming a solution to Eq. (†) of the form

$$y(x) = \sum_{n=1}^{\infty} a_n x^n + (b + cx)\ln x \quad ,$$

where all coefficients are real, determine the value of c. [4]

(d) Determine a_1, a_2, and the recursion relation between a_n and a_{n-1} whenever $n \geqslant 3$. [8]

(e) For what range of x values does this solution converge? [1]

5A You wish to have a fence built in the (x, y)-plane, joining the points $(-a, 0)$ and $(a, 0)$. You are required by law to ensure that the fence follows a single-valued curve $y(x)$ for $-a \leqslant x \leqslant a$ and has no points with x coordinates outside this range.

Company A will build a fence of any length, at a cost of

$$\int_{-a}^{a} dx \left(K + \dot{y}^2(x) + y^2(x) - 2xy(x)\right) ,$$

where $\dot{y} = \frac{dy}{dx}$ and K is a positive constant.

Company B will only build a fence of length L (which is fixed by them and obeys $\pi a \geqslant L > 2a$) and only in the lower half plane and will charge

$$\int_{-a}^{a} dx \left(K' + y(x)\right) ,$$

where K' is a positive constant.

Company C will also only build a fence of length L (which is similarly fixed by them and obeys $\pi a \geqslant L > 2a$) and only in the lower half plane at a cost of

$$\int_{-a}^{a} dx \left(K'' + y(x)\left(1 + \dot{y}^2(x)\right)^{\frac{1}{2}}\right) ,$$

where K'' is a positive constant.

(The constants K, K', K'' are large enough that any fence built by the relevant company will have positive cost.)

Considering each company separately, give equations characterising the curve $y(x)$ you would choose if you employ that company, assuming in each case that you wish to minimize the cost. In the cases of companies B and C, determine the relationship between the parameters in your solutions and the fence length L. Justify your answers (in all three cases) with derivations. [6,6,6]

Sketch the curves $y(x)$ that you have obtained. [2]

6B

(a) Consider a second-order differential operator $\tilde{\mathcal{L}}$ defined for x in the range $a \leqslant x \leqslant b$ by

$$\tilde{\mathcal{L}}y = p(x)y'' + r(x)y' + q(x)y\,, \qquad (\dagger)$$

where $y(x)$, $p(x)$, $r(x)$, $q(x)$ are real-valued functions, $p(x) > 0$ for x in the range $a < x < b$, y satisfies specified boundary conditions at $x = a$ and $x = b$, and the prime denotes differentiation with respect to x.

(i) With the inner product

$$\langle f, g\rangle = \int_a^b f\,g\,dx\,,$$

$\tilde{\mathcal{L}}$ is self-adjoint if $\langle y_1, \tilde{\mathcal{L}}y_2\rangle = \langle \tilde{\mathcal{L}}y_1, y_2\rangle$ for any two functions y_1, y_2 satisfying the boundary conditions. Assuming that $r = p'$, determine for which boundary conditions the operator $\tilde{\mathcal{L}}$ defined in Eq. ($\dagger$) is self-adjoint. [5]

(ii) The function y is an eigenfunction of $\tilde{\mathcal{L}}$ with eigenvalue λ if $\tilde{\mathcal{L}}y = \lambda y$. If $r \neq p'$, we can define the weight function

$$w(x) = \exp\left(\int_a^x \frac{r(u) - p'(u)}{p(u)}\,du\right)\,,$$

and rewrite the differential equation $\tilde{\mathcal{L}}y = \lambda y$ in the form $\mathcal{L}y = \lambda w y$, where $\mathcal{L} = w\tilde{\mathcal{L}}$ and $\mathcal{L}$ is a Sturm-Liouville operator. Now consider Bessel's equation, which is given by

$$\tilde{\mathcal{L}}y = x^2y'' + xy' + (\beta^2x^2 - n^2)y = 0\,,$$

where $x \geqslant 0$ and β, n are real constants. Using the above method, show that Bessel's equation can be written as an eigenvalue equation for a differential operator of Sturm-Liouville form with eigenvalue n^2.

[6]

(b) Consider the inhomogeneous Sturm-Liouville equation

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y - \lambda w(x)y = w(x)f(x)\,, \qquad (\ddagger)$$

for x in the range $a \leqslant x \leqslant b$. Here p and q are real functions on this range, w is the weight function and solutions satisfy boundary conditions of the form $\alpha y' + \beta y = 0$ at $x = a$ and $\kappa y' + \mu y = 0$ at $x = b$ (where at least one of $\alpha,\ \beta$ and at least one of $\kappa,\ \mu$ are nonzero). Let $G(x,\xi)$ be the Green's function satisfying

$$\frac{d}{dx}\left(p(x)\frac{dG}{dx}\right) + q(x)G(x,\xi) = \delta(x-\xi)\,,$$

with the same boundary conditions as y for the same constants $\alpha,\ \beta,\ \kappa,\ \mu$. Show that for these boundary conditions the Sturm-Liouville equation ($\ddagger$) can be written as the integral equation

$$\int_a^b f(x)w(x)G(x,\xi)dx + \lambda\int_a^b w(x)G(x,\xi)y(x)dx - y(\xi) = 0\,.$$

[9]

7C

(a) For a function $f(z)$ analytic in a region containing a closed contour C and a point z_0 within the contour, Cauchy's formula states that

$$\left.\frac{d^n f(z)}{dz^n}\right|_{z=z_0} = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz \quad .$$

Using this formula evaluate the integrals

$$\oint_C \frac{\sin(z)}{z^{n+1}}dz \qquad \text{and} \qquad \oint_C \frac{\cos(z)}{z^{n+1}}dz \quad ,$$

where C is a counterclockwise contour of radius $R > 0$ centred at the origin and n is a positive integer.

[6]

(b) For a function

$$f(z) = \frac{g(z)}{z^2 - z_0^2} \quad ,$$

where $g(z)$ is analytic everywhere in the complex plane and $z_0 \neq 0$, identify the poles of $f(z)$ and their residues. [2]

(c) State the residue theorem.

[2]

(d) Using the residue theorem show that, for n a positive even integer,

$$\int_{-\infty}^{\infty} \frac{1}{1+x^n}dx = \frac{A_n}{\sin(B_n)} \quad ,$$

where A_n and B_n are n-dependent constants which you should determine.

[*Hint: L'Hôpital's rule and standard formulae for geometric summation may prove useful.*]

[10]

8B

(a) The Fourier transform and its inverse of an absolutely integrable function $f(t)$ with bounded variation and a finite number of discontinuities are given by

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt\,, \qquad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)e^{i\omega t}d\omega\,.$$

(i) Compute the Fourier transform of df/dt. [2]

(ii) Show that for a real valued function $f(t)$, the Fourier transform satisfies the condition $[\tilde{f}(-\omega)]^* = \tilde{f}(\omega)$, where the * denotes the complex conjugate. [3]

(b) The massive wave equation in one spatial dimension for a function $\sigma(t,r)$ is

$$\partial_t^2\sigma - \partial_r^2\sigma + \mu^2\sigma = 0\,, \qquad (\dagger)$$

where $\partial_r = \partial/\partial r$, $\partial_t = \partial/\partial t$, $-\infty < t < \infty$, $0 \leqslant r < \infty$ and $\mu > 0$ is a constant.

(i) Applying a Fourier transform in the time domain, show that Eq. ($\dagger$) can be written as a differential equation for $\tilde{\sigma}(\omega,r)$,

$$\partial_r^2\tilde{\sigma} = (\mu^2 - \omega^2)\tilde{\sigma}\,.$$

[1]

(ii) Show that the solution to Eq. ($\dagger$) is given by

$$\sigma(t,r) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[A(\omega)e^{ikr} + B(\omega)e^{-ikr}\right]e^{i\omega t}\,d\omega\,,$$

where k is a function of ω which you should determine, and A and B are undetermined functions. [4]

(iii) Now we require that $\sigma(t,r)$ is a real valued function that is bounded on the range $0 \leqslant r < \infty$. Derive from these requirements one constraint on the functions $A(\omega)$ and $B(\omega)$ that holds for $|\omega| > \mu$, and two independent constraints on $A(\omega)$, $B(\omega)$ that hold for $|\omega| < \mu$. [5]

(iv) Using the results of part (iii), show that

$$\sigma(t,r) = \lambda\ \mathrm{Re}\left[\int_0^{\mu} A(\omega)e^{ikr}e^{i\omega t}d\omega + \int_{\mu}^{\infty}\left[B(\omega)e^{-ikr}e^{i\omega t} + B^*(-\omega)e^{ikr}e^{i\omega t}\right]d\omega\right]$$

is a solution of the massive wave equation ($\dagger$). Here Re[...] denotes the real part and λ is a constant you should determine. [5]

9A

[*Hints: In this question, you are not required to solve the characteristic equation to find the eigenvalues and modes from the eigenvalue equation. You may find it helpful to consider appropriate orthogonality conditions for the modes.*]

Small spheres of mass m are linked by springs of spring constant k in deep space, where the gravitational potential is negligible. They are arranged so that at equilibrium the masses form the six vertices of a regular octahedron, whose coordinates are $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$ in a suitable Cartesian frame, while the springs lie along the edges of the octahedron (so there are twelve springs in total).

(a) Define generalized coordinates by small displacements from this equilibrium. Show that there are at least six independent zero modes, giving these modes explicitly in the given coordinates, and describe them geometrically. [5]

(b) Explain why the system must have a "dilation" (or "breathing") mode of oscillation in which at each time all the masses are equally distant from the origin. Determine its frequency of oscillation. [6]

(c) At time 0 the system is displaced so that the masses are held stationary at coordinates $(\pm(1+\epsilon_1), 0, 0)$, $(0, \pm(1+\epsilon_2), 0)$, $(0, 0, \pm(1+\epsilon_2))$, where ϵ_1 and ϵ_2 are small. They are then released. Give an equation for the state of the system at times $t > 0$. [9]

10A

(a) Explain how the *alternating group* A_n can be defined as a subgroup of the permutation group S_n. Define the *cycle shape* of a permutation. How many elements does A_4 have and what are their cycle shapes? [4]

(b) Show that A_4 is isomorphic to the group of rotation symmetries of a rigid regular tetrahedron. Hence show that A_4 has an irreducible three-dimensional representation. [You may assume without proof that a regular tetrahedron has rotational symmetries about precisely seven axes, which you should identify.] [8]

(c) Determine the dimensions of all irreducible representations of A_4. Use this result to determine whether all pairs of elements of A_4 with the same cycle shape belong to the same conjugacy class.

[*You may quote without proof any relevant theorems, provided they are clearly stated.*] [8]

END OF PAPER