UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Monday, 8 June, 2020

MATHEMATICS FORMATIVE ASSESSMENT

Before you begin read these instructions carefully:

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

You may upload your answers to ***all*** *of section A, and to no more than* ***five*** *questions from section B. For this purpose, your section A attempts should be considered as one single answer, and must be uploaded together in a single file. You may upload one file each for your five section B answers, making six files in all.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Calculators are not permitted in this examination. This is a closed-book examination, which you should do under self-imposed examination conditions.

SECTION A

1

Given that $(x-2)$ is a factor of (x^3-x^2-x-2) find the other factor. [2]

2

Solve for θ in the interval $\frac{\pi}{2}<\theta<\frac{3\pi}{2}$:

$$(\sin\theta-\cos\theta)(\sin^2\theta+2\cos^2\theta)\leqslant 0.$$

[2]

3

For $x\neq 0$, differentiate $y=x\sin\frac{1}{x}$. [2]

4

Consider the curves $x^2+2y^2=1$ and $ax^2-y^2=1$ where x, y and the constant a are real, and $a>0$. For what values of a do the curves have no shared points? [2]

5

Given $\tan\dfrac{\pi}{4}=1$, $\tan\dfrac{\pi}{8}=t$, and using the formula for $\tan 2\theta$ in terms of $\tan\theta$:

(a) Find a quadratic equation for t. [1]

(b) Solve this equation to find $\tan\dfrac{\pi}{8}$. [1]

6

Calculate the indefinite integral $\int x\sqrt{3-2x}\,dx$ where $x\leqslant 3/2$. [2]

7

Find the angle between the line segments joining the origin $(0,0,0)$ to the points $(1,0,-1)$ and $(0,1,1)$. [2]

8

For positive x, sketch the graph of $y=\dfrac{1}{x^2}-\dfrac{1}{x}$ and mark the value of x where y is stationary.

[2]

9

Write down an explicit formula for the sum to one hundred terms of the geometric series with common ratio r and first term ar. For what values of a and r is your formula valid? [2]

10

The recurrence relation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

where $n = 0, 1, 2, \ldots$ is used to approximate the root of $f(x) = x^2 - 2$ for $x > 0$. Taking $x_0 = 1$, calculate x_1, and then calculate x_2 to two decimal places. [2]

SECTION B

11W

(a) Find the real and imaginary parts of the following complex numbers:

(i) $(2-i)^{-1}$; [2]

(ii) $\exp\left(\sum_{n=1}^{45} i^n\right)$; [2]

(iii) $\ln\left[\sinh\left(\frac{i\pi}{2}\right)+\cosh\left(\frac{9i\pi}{2}\right)\right]$; [3]

(iv) $\sum_{n=1}^{121}\left[\tanh\left(\frac{in\pi}{4}\right)-\tanh\left(\frac{in\pi}{4}-\frac{i\pi}{4}\right)\right]$. [3]

(b) Find the locus that solves $|z^*+2i|+|z|=a$, and sketch it on an Argand diagram for

(i) $a=4$, [6]

(ii) $a=2$. [4]

[*z^* indicates the complex conjugate of z.*]

12R

(a) Write down the area element dS in polar coordinates (r,θ), and use it to [1]

(i) Calculate $\int_{-\infty}^{+\infty} e^{-x^2}\,dx$. [2]

(ii) Calculate the area enclosed by the curve $r=a(1+\cos\theta), 0\leqslant\theta<2\pi$, where $a>0$. [3]

(b) Give the expression for the volume element dV in Cartesian, cylindrical and spherical coordinates. [3]

(i) Let P be the volume of an infinite paraboloid defined by $z\geqslant x^2+y^2$. Calculate

$$\iiint_P f'(z)dV \qquad \text{where} \quad f(z)=-\frac{e^{-z}}{\sqrt{z}}.$$

[5]

(ii) Calculate the volume of intersection between the unit ball, defined by

$$x^2+y^2+z^2\leqslant 1\,,$$

and the solid cone, defined by $\sqrt{x^2+y^2}\leqslant z\tan\alpha$, both using spherical coordinates and using cylindrical coordinates. Assume $0<\alpha<\pi/2$. [6]

13Y

(a) Consider the equation

$$\frac{dy}{dx} + \frac{y}{\tan(x)} = \cos^2(x), \quad 0 < x < \pi/2\,.$$

(i) Find an integrating factor. [3]

(ii) Find the general solution. [3]

(b) A differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

can be written as $\hat{L}y = f(x)$ where $\hat{L}$ is a linear differential operator defined as

$$\hat{L} = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c\,.$$

Consider a linear differential operator of the form

$$\hat{L} = \frac{d^2}{dx^2} + \sqrt{3}\frac{d}{dx} + 3\,.$$

(i) Find the general solution $y_c(x)$ of $\hat{L}y = 0$. [4]

(ii) Find a particular solution $y_{p,1}(x)$ of $\hat{L}y = e^{-\sqrt{3}x}$. [3]

(iii) Find a particular solution $y_{p,2}(x)$ of $\hat{L}y = x$. [3]

(iv) Find the general solution $y(x)$ of $\hat{L}y = 2x + e^{-\sqrt{3}x}$ with boundary conditions

$$y(0) = 0 \quad \text{and} \quad y(\pi) = \frac{e^{-\sqrt{3}\pi}}{3} - \frac{2}{3\sqrt{3}}\,.$$

[4]

14Q

(a) Find the locations of, and the values of f at, the stationary points of the function

$$f(x,y) = (1 - x^2 - y^2)^2 + \frac{7}{2}\ln(3 + y^2).$$

[6]

(b) Determine the natures of the stationary points. [4]

(c) Plot the contours of the function. [6]

(d) Add arrows to your diagram to show the direction of ∇f at illustrative points close to the stationary points and on the lines $x = 0$ and $y = 0$. [4]

15V

(a) State Taylor's theorem by giving the series expansion about $x = a$ of a function $f(x)$ that is n times differentiable, showing the first n terms, together with an expression for the remainder term R_n. [4]

(b) Taking $f(x) = x^{1/5}$, find an approximation for $(33)^{1/5}$ by using the first three terms of the Taylor series expansion of $f(x)$ about $a = 32$. [4]

(c) Find the Taylor series expansions about a, up to and including term in $(x-a)^4$, of

(i) $\frac{\sin x}{\sinh x}$ for $a = 0$, [6]

(ii) $\sinh(\ln x)$ for $a = 1$. [6]

[*You may quote and use standard Taylor series expansions of known functions.*]

16V

(a) I bought 5 apples and 5 bananas to make 10 packed lunches (sequentially), each containing one fruit drawn at random. Let $P(A_i)$ denote the probability of putting an apple into the i-th packed lunch, and $P(B_i)$ denote the probability of putting a banana into the i-th packed lunch $(1 \leqslant i \leqslant 10)$.

(i) Find $P(A_1)$ and $P(B_2)$. [3]

(ii) Find $P(A_3|A_2)$. [3]

(iii) Use Bayes' theorem to compute $P(A_2|A_3)$. [2]

(iv) Find $P(B_1 \cup A_2)$. [2]

(v) Find the expected number of apples remaining after making the first three packed lunches. [4]

(b) Pieces of fruit are rotten independently with probability p.

(i) From my 10 pieces of fruit, what is the probability that exactly n are rotten $(0 \leqslant n \leqslant 10)$? [2]

(ii) Find the standard deviation of the number of rotten fruit. [4]

17S

(a) Evaluate the indefinite integrals for real x:

(i) $\displaystyle\int \frac{\sin x}{\cos^2 x - 5\cos x + 6}\, dx\,,$ [4]

(ii) $\displaystyle\int \frac{\ln x}{x^4}\, dx$ for $x > 0\,,$ [4]

(iii) $\displaystyle\int \sqrt{1-x^2}\, dx$ for $|x| \leqslant 1\,.$ [4]

(b) Show, by means of a suitable substitution, that the integral

$$I = \int_{\arccos(c)}^{\frac{\pi}{2}} \tan(x)\cos^4(x)\, dx \qquad \text{with } 0 < c < 1,$$

is equal to

$$J = \int_0^c t^3 dt\,.$$

Give the value of J, and verify that this is the same value as the one obtained by defining the integral J as the limit of a sum. [8]

$$\left[\textit{Hint: It can be assumed that } \sum_{k=1}^{n} k^3 = \frac{n^2(1+n)^2}{4}. \right]$$

18Z

(a) Given the matrix

$$\boldsymbol{A} = \begin{pmatrix} a & 0 & c-1 \\ 0 & b & 0 \\ c-1 & 0 & a \end{pmatrix},$$

find all possible values for the real paramters a, b and c for which $\boldsymbol{A}$ is

(i) an orthogonal matrix; [4]

(ii) a rotation matrix. [4]

You need to present your answers in the form $(a, b, c)^{\mathrm{T}} = (p_1, p_2, p_3)^{\mathrm{T}}$ where the values of p_1, p_2 and p_3 should be determined.

(b) Given the matrix

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2-\lambda & 2(1-\lambda) \\ 0 & 3 & 3 \\ -\lambda & 9-\lambda & 18-\lambda \end{pmatrix},$$

find the values of the real parameter λ for which $\boldsymbol{A}\boldsymbol{x} = 0$ (where $\boldsymbol{x}^{\mathrm{T}} = (x, y, z)$) has non-trivial solutions. [2]

(c) Given the matrix $\boldsymbol{A} = \begin{pmatrix} 4 & 3 \\ 3 & -4 \end{pmatrix}$,

(i) find two matrices $\boldsymbol{B}$ and $\boldsymbol{C}$ such that $\boldsymbol{A} = \boldsymbol{B}\boldsymbol{C}\boldsymbol{B}^{\mathrm{T}}$, where $\boldsymbol{B}$ is orthogonal and anti-symmetric; [5]

(ii) find the eigenvalues and normalised eigenvectors of $\boldsymbol{C}$. [5]

19X

In this question all vectors are given in Cartesian form, that is:

$$\boldsymbol{F} = (F_x, F_y, F_z) = F_x\boldsymbol{i} + F_y\boldsymbol{j} + F_z\boldsymbol{k}.$$

(a) For the scalar field $\psi = \frac{1}{2}(x^2 + y^2 + z^2)$ calculate $\nabla^2\psi$. [2]

(b) The open surface C is defined as the curved surface of the cone whose apex is the point $(0, 0, b)$ with $b > 0$, and whose base is the circle in the plane $z = 0$ with centre at the origin and radius a. Show, with the aid of a diagram or otherwise, that $d\boldsymbol{S}$, the vector element of area on C, is given by the vector product of the two vectors:

$$(\cos\theta, \sin\theta, -b/a)dr \quad \text{and} \quad r(-\sin\theta, \cos\theta, 0)d\theta$$

where r, θ and z are the usual cylindrical polar coordinates. Hence calculate $d\boldsymbol{S}$. [6]

(c) Without using the divergence theorem, for the vector field $\boldsymbol{F} = (x, y, z)$ calculate the flux of $\boldsymbol{F}$ through C. [6]

(d) Calculate the flux Φ of $\boldsymbol{F} = (x, y, z)$ through the closed spherical surface of radius a centred at the origin. [3]

(e) Let the open surface C' represent the curved surface of the cone whose apex is the point $(0, b, 0)$ with $b > 0$ and whose base is the circle in the plane $y = 0$ with centre at the origin and radius a. Find the flux of $\boldsymbol{F} = (x, y, z)$ through C'. [3]

20S

Consider the vector fields $\boldsymbol{F}$ and $\boldsymbol{G}$

$$\boldsymbol{F}(\mathbf{x}) = \begin{pmatrix} y+z \\ x+z \\ x+y \end{pmatrix}, \qquad \boldsymbol{G}(\mathbf{x}) = \begin{pmatrix} y^2+z^2 \\ x^2+z^2 \\ x^2+y^2 \end{pmatrix}.$$

(a) Evaluate $\int \boldsymbol{F}\cdot d\mathbf{x}$ and $\int \boldsymbol{G}\cdot d\mathbf{x}$ along the straight line from (1,0,0) to (0,1,1). [4]

(b) Show that the integrals $\int \boldsymbol{F}\cdot d\mathbf{x}$ and $\int \boldsymbol{G}\cdot d\mathbf{x}$ are unchanged if the path is taken to be $\mathbf{x}(t) = (1 - t^n, t^n, t^n)$, with $0 \leqslant t \leqslant 1$, and $n > 1$. [5]

(c) Show that $\boldsymbol{F}$ is a conservative field by computing $\nabla \times \mathbf{F}$ and find the general scalar field $\Phi(\mathbf{x})$ such that $\boldsymbol{F} = \nabla\Phi$. [4]

(d) Show that $\boldsymbol{G}$ is not a conservative field. [3]

(e) Find a path $\mathbf{x}(t)$ such that the line integral $\int \boldsymbol{G}\cdot d\mathbf{x}$ from (1,0,0) to (0,1,1) yields a different result compared to the one found in (a). [4]

END OF PAPER