MATHEMATICS FORMATIVE ASSESSMENT

Before you begin read these instructions carefully:

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

You may upload your answers to all of section A, and to no more than five questions from section B. For this purpose, your section A attempts should be considered as one single answer, and must be uploaded together in a single file. You may upload one file each for your five section B answers, making six files in all.

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Calculators are not permitted in this examination. This is a closed-book examination, which you should do under self-imposed examination conditions.
SECTION A

1
Given that \((x - 2)\) is a factor of \((x^3 - x^2 - x - 2)\) find the other factor. [2]

2
Solve for \(\theta\) in the interval \(\frac{\pi}{2} < \theta < \frac{3\pi}{2}\):
\[
(sin \theta - cos \theta)(sin^2 \theta + 2 cos^2 \theta) \leq 0.
\] [2]

3
For \(x \neq 0\), differentiate \(y = x \sin \frac{1}{x}\). [2]

4
Consider the curves \(x^2 + 2y^2 = 1\) and \(ax^2 - y^2 = 1\) where \(x\), \(y\) and the constant \(a\) are real, and \(a > 0\). For what values of \(a\) do the curves have no shared points? [2]

5
Given \(tan \frac{\pi}{4} = 1\), \(tan \frac{\pi}{8} = t\), and using the formula for \(tan 2\theta\) in terms of \(tan \theta\):
(a) Find a quadratic equation for \(t\). [1]
(b) Solve this equation to find \(tan \frac{\pi}{8}\). [1]

6
Calculate the indefinite integral \(\int x \sqrt{3 - 2x} \, dx\) where \(x \leq 3/2\). [2]

7
Find the angle between the line segments joining the origin \((0,0,0)\) to the points \((1,0,-1)\) and \((0,1,1)\). [2]

8
For positive \(x\), sketch the graph of \(y = \frac{1}{x^2} - \frac{1}{x}\) and mark the value of \(x\) where \(y\) is stationary. [2]
9

Write down an explicit formula for the sum to one hundred terms of the geometric series with common ratio $r$ and first term $ar$. For what values of $a$ and $r$ is your formula valid? [2]

10

The recurrence relation
\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]
where $n = 0, 1, 2, \ldots$ is used to approximate the root of $f(x) = x^2 - 2$ for $x > 0$. Taking $x_0 = 1$, calculate $x_1$, and then calculate $x_2$ to two decimal places. [2]
SECTION B

11W

(a) Find the real and imaginary parts of the following complex numbers:

(i) \((2 - i)^{-1}\); \[2\]
(ii) \(\exp\left(\sum_{n=1}^{45} i^n\right)\); \[2\]
(iii) \(\ln\left[\sinh\left(\frac{i\pi}{2}\right) + \cosh\left(\frac{9i\pi}{2}\right)\right]\); \[3\]
(iv) \(\sum_{n=1}^{121} \left[\tanh\left(\frac{in\pi}{4}\right) - \tanh\left(\frac{in\pi}{4} - \frac{i\pi}{4}\right)\right]\). \[3\]

(b) Find the locus that solves \(|z^* + 2i| + |z| = a\), and sketch it on an Argand diagram for

(i) \(a = 4\), \[6\]
(ii) \(a = 2\). \[4\]

\([z^* \text{ indicates the complex conjugate of } z.\)

12R

(a) Write down the area element \(dS\) in polar coordinates \((r, \theta)\), and use it to

(i) Calculate \(\int_{-\infty}^{+\infty} e^{-x^2} \, dx\). \[2\]
(ii) Calculate the area enclosed by the curve \(r = a(1 + \cos \theta), 0 \leq \theta < 2\pi\), where \(a > 0\). \[3\]

(b) Give the expression for the volume element \(dV\) in Cartesian, cylindrical and spherical coordinates.

(i) Let \(P\) be the volume of an infinite paraboloid defined by \(z \geq x^2 + y^2\). Calculate

\(\iiint_P f'(z) \, dV\) where \(f(z) = -\frac{e^{-z}}{\sqrt{z}}\). \[5\]

(ii) Calculate the volume of intersection between the unit ball, defined by \(x^2 + y^2 + z^2 \leq 1\),

and the solid cone, defined by \(\sqrt{x^2 + y^2} \leq z \tan \alpha\), both using spherical coordinates and using cylindrical coordinates. Assume \(0 < \alpha < \pi/2\). \[6\]
13Y

(a) Consider the equation
\[ \frac{dy}{dx} + \frac{y}{\tan(x)} = \cos^2(x), \quad 0 < x < \pi/2. \]

(i) Find an integrating factor.  \[3\]
(ii) Find the general solution.  \[3\]

(b) A differential equation
\[ a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = f(x) \]
can be written as \( \hat{L}y = f(x) \) where \( \hat{L} \) is a linear differential operator defined as
\[ \hat{L} = a \frac{d^2}{dx^2} + b \frac{d}{dx} + c. \]
Consider a linear differential operator of the form
\[ \hat{L} = \frac{d^2}{dx^2} + \sqrt{3} \frac{d}{dx} + 3. \]

(i) Find the general solution \( y_c(x) \) of \( \hat{L}y = 0 \).  \[4\]
(ii) Find a particular solution \( y_{p,1}(x) \) of \( \hat{L}y = e^{-\sqrt{3}x} \).  \[3\]
(iii) Find a particular solution \( y_{p,2}(x) \) of \( \hat{L}y = x \).  \[3\]
(iv) Find the general solution \( y(x) \) of \( \hat{L}y = 2x + e^{-\sqrt{3}x} \) with boundary conditions
\[ y(0) = 0 \quad \text{and} \quad y(\pi) = \frac{e^{-\sqrt{3}\pi}}{3} - \frac{2}{3\sqrt{3}}. \]  \[4\]

14Q

(a) Find the locations of, and the values of \( f \) at, the stationary points of the function
\[ f(x, y) = (1 - x^2 - y^2)^2 + \frac{7}{2} \ln(3 + y^2). \]  \[6\]

(b) Determine the natures of the stationary points.  \[4\]
(c) Plot the contours of the function.  \[6\]
(d) Add arrows to your diagram to show the direction of \( \nabla f \) at illustrative points close to the stationary points and on the lines \( x = 0 \) and \( y = 0 \).  \[4\]

Natural Sciences IA, Mathematics formative assessment
15V

(a) State Taylor’s theorem by giving the series expansion about \( x = a \) of a function \( f(x) \) that is \( n \) times differentiable, showing the first \( n \) terms, together with an expression for the remainder term \( R_n \). [4]

(b) Taking \( f(x) = x^{1/5} \), find an approximation for \( (33)^{1/5} \) by using the first three terms of the Taylor series expansion of \( f(x) \) about \( a = 32 \). [4]

(c) Find the Taylor series expansions about \( a \), up to and including term in \( (x - a)^4 \), of

(i) \( \frac{\sin x}{\sinh x} \) for \( a = 0 \), [6]

(ii) \( \sinh(\ln x) \) for \( a = 1 \). [6]

[You may quote and use standard Taylor series expansions of known functions.]

16V

(a) I bought 5 apples and 5 bananas to make 10 packed lunches (sequentially), each containing one fruit drawn at random. Let \( P(A_i) \) denote the probability of putting an apple into the \( i \)-th packed lunch, and \( P(B_i) \) denote the probability of putting a banana into the \( i \)-th packed lunch (1 \( \leq \) \( i \) \( \leq \) 10).

(i) Find \( P(A_1) \) and \( P(B_2) \). [3]

(ii) Find \( P(A_3|A_2) \). [3]

(iii) Use Bayes’ theorem to compute \( P(A_2|A_3) \). [2]

(iv) Find \( P(B_1 \cup A_2) \). [2]

(v) Find the expected number of apples remaining after making the first three packed lunches. [4]

(b) Pieces of fruit are rotten independently with probability \( p \).

(i) From my 10 pieces of fruit, what is the probability that exactly \( n \) are rotten (0 \( \leq \) \( n \) \( \leq \) 10)? [2]

(ii) Find the standard deviation of the number of rotten fruit. [4]
(a) Evaluate the indefinite integrals for real $x$:

(i) \[ \int \frac{\sin x}{\cos^2 x - 5 \cos x + 6} \, dx , \quad [4] \]

(ii) \[ \int \frac{\ln x}{x^4} \, dx \quad \text{for } x > 0 , \quad [4] \]

(iii) \[ \int \sqrt{1 - x^2} \, dx \quad \text{for } |x| \leq 1 . \quad [4] \]

(b) Show, by means of a suitable substitution, that the integral

\[ I = \int_{\arccos(c)}^{\frac{\pi}{2}} \tan(x) \cos^4(x) \, dx \quad \text{with } 0 < c < 1 , \]

is equal to

\[ J = \int_0^c t^3 \, dt . \]

Give the value of $J$, and verify that this is the same value as the one obtained by defining the integral $J$ as the limit of a sum. \([8]\]

\[ \text{Hint: It can be assumed that } \sum_{k=1}^{n} k^3 = \frac{n^2(1 + n)^2}{4}. \]
(a) Given the matrix
\[ A = \begin{pmatrix} a & 0 & c - 1 \\ 0 & b & 0 \\ c - 1 & 0 & a \end{pmatrix}, \]
find all possible values for the real parameters \(a, b\) and \(c\) for which \(A\) is

(i) an orthogonal matrix;
(ii) a rotation matrix.

You need to present your answers in the form \((a, b, c)^T = (p_1, p_2, p_3)^T\) where the values of \(p_1, p_2\) and \(p_3\) should be determined.

(b) Given the matrix
\[ A = \begin{pmatrix} 1 & 2 - \lambda & 2(1 - \lambda) \\ 0 & 3 & 3 \\ -\lambda & 9 - \lambda & 18 - \lambda \end{pmatrix}, \]
find the values of the real parameter \(\lambda\) for which \(Ax = 0\) (where \(x^T = (x, y, z)\)) has non-trivial solutions.

(c) Given the matrix \(A = \begin{pmatrix} 4 & 3 \\ 3 & -4 \end{pmatrix}\),

(i) find two matrices \(B\) and \(C\) such that \(A = BCB^T\), where \(B\) is orthogonal and anti-symmetric;
(ii) find the eigenvalues and normalised eigenvectors of \(C\).
In this question all vectors are given in Cartesian form, that is:
\[ \mathbf{F} = (F_x, F_y, F_z) = F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k}. \]

(a) For the scalar field \( \psi = \frac{1}{2}(x^2 + y^2 + z^2) \) calculate \( \nabla^2 \psi \). \[2\]

(b) The open surface \( C \) is defined as the curved surface of the cone whose apex is the point (0, 0, b) with \( b > 0 \), and whose base is the circle in the plane \( z = 0 \) with centre at the origin and radius \( a \). Show, with the aid of a diagram or otherwise, that \( dS \), the vector element of area on \( C \), is given by the vector product of the two vectors:
\[
\begin{align*}
& (\cos \theta, \sin \theta, -b/a) \, dr \quad \text{and} \quad r(-\sin \theta, \cos \theta, 0) \, d\theta \\
& \text{where } r, \theta \text{ and } z \text{ are the usual cylindrical polar coordinates. Hence calculate } dS. \[6\]
\]

(c) Without using the divergence theorem, for the vector field \( \mathbf{F} = (x, y, z) \) calculate the flux of \( \mathbf{F} \) through \( C \). \[6\]

(d) Calculate the flux \( \Phi \) of \( \mathbf{F} = (x, y, z) \) through the closed spherical surface of radius \( a \) centred at the origin. \[3\]

(e) Let the open surface \( C' \) represent the curved surface of the cone whose apex is the point (0, b, 0) with \( b > 0 \) and whose base is the circle in the plane \( y = 0 \) with centre at the origin and radius \( a \). Find the flux of \( \mathbf{F} = (x, y, z) \) through \( C' \). \[3\]

Consider the vector fields \( \mathbf{F} \) and \( \mathbf{G} \)
\[
\begin{align*}
& \mathbf{F}(x) = \begin{pmatrix} y + z \\ x + z \\ x + y \end{pmatrix}, & & \mathbf{G}(x) = \begin{pmatrix} y^2 + z^2 \\ x^2 + z^2 \\ x^2 + y^2 \end{pmatrix}.
\end{align*}
\]

(a) Evaluate \( \int \mathbf{F} \cdot d\mathbf{x} \) and \( \int \mathbf{G} \cdot d\mathbf{x} \) along the straight line from (1,0,0) to (0,1,1). \[4\]

(b) Show that the integrals \( \int \mathbf{F} \cdot d\mathbf{x} \) and \( \int \mathbf{G} \cdot d\mathbf{x} \) are unchanged if the path is taken to be \( \mathbf{x}(t) = (1 - t^n, t^n, t^n) \), with \( 0 \leq t \leq 1 \), and \( n > 1 \). \[5\]

(c) Show that \( \mathbf{F} \) is a conservative field by computing \( \nabla \times \mathbf{F} \) and find the general scalar field \( \Phi(\mathbf{x}) \) such that \( \mathbf{F} = \nabla \Phi \). \[4\]

(d) Show that \( \mathbf{G} \) is not a conservative field. \[3\]

(e) Find a path \( \mathbf{x}(t) \) such that the line integral \( \int \mathbf{G} \cdot d\mathbf{x} \) from (1,0,0) to (0,1,1) yields a different result compared to the one found in (a). \[4\]