UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB

Friday, 31 May, 2019 9:00 am to 12:00 pm NST1

MATHEMATICS (2)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, ***6B****).*

Answers must be tied up in ***separate*** *bundles, marked* ***A, B or C*** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet ***must*** *be completed and attached to the bundle.*

A ***separate*** *green master cover sheet listing all the questions attempted* ***must*** *also be completed.*

Every cover sheet must bear your examination number and desk number.

Calculators are not permitted in this examination.

STATIONERY REQUIREMENTS
3 blue cover sheets and treasury tags
Green master cover sheet
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1A

(a) Give the general form of a second-order differential operator in Sturm-Liouville form. [2]

(b) Consider the eigenvalue equation

$$-xy'' - (1-x)y' = ny, \qquad 0 \leqslant x < \infty, \tag{1}$$

where n is a non-negative integer. Find a weight function $w(x)$ to put (1) into Sturm-Liouville form. The solution to (1) is a real polynomial p_n of degree n. Using the normalization $p_n(0) = 1$, compute the polynomials p_0, p_1, p_2 and verify that $\|p_0\|_w^2 = \|p_1\|_w^2 = \|p_2\|_w^2 = 1$, where $\|p\|_w^2 = \int_0^\infty w(x)p(x)^2 dx$. [8]

(c) Let f be an arbitrary function such that $\|f\|_w^2$ is finite. We are interested in finding the best approximation of f that minimizes

$$\left\| f - \sum_{n=0}^{N} a_n p_n \right\|_w^2 \tag{2}$$

for a given integer N and real coefficients a_n. Give an expression for the optimal choice of $a_0, \ldots, a_N$ that minimizes (2). Compute these values for $f(x) = e^{-x}(1+x)$ and $N = 2$. [5]

(d) Prove that the polynomials p_n satisfy $\|p_n\|_w^2 = 1$ for all n, assuming the normalization $p_n(0) = 1$. [*Hint: You can use the fact that any polynomial of degree d has an expansion $\sum_{n=0}^{d} c_n p_n$ for some coefficients $c_0, \ldots, c_d$.*] [5]

2A

(a) Consider Laplace's equation in spherical polar coordinates when $\Psi(r, \theta, \phi)$ is axisymmetric:

$$\nabla^2 \Psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Psi}{\partial \theta} \right) = 0.$$

Using separation of variables show that the general solution takes the form

$$\Psi(r, \theta) = \sum_{\ell=0}^{\infty} \left(A_\ell \, r^\ell + B_\ell \, r^{-\ell-1} \right) P_\ell(\cos \theta),$$

where $P_\ell(u)$ is the Legendre polynomial of degree ℓ, the solution of the Legendre equation $\frac{d}{du} \left[(1-u^2) \frac{dP_\ell}{du} \right] + \lambda P_\ell(u) = 0$ with $\lambda = \ell(\ell+1)$. [*Hint: You may assume without proof that the Legendre equation has solutions which are well-behaved at* $u = \pm 1$ *only for such choices of* λ.] [10]

(b) Assuming the boundary condition $\Psi = 1 + 2\cos\theta - 3\sin^2\theta$ on the surface of a sphere of radius a, find Ψ inside and outside the sphere assuming that Ψ is finite everywhere and $\Psi \to 0$ as $r \to \infty$. [*Hint: You may use the fact that* $P_0(u) = 1$, $P_1(u) = u$, *and* $P_2(u) = (3u^2 - 1)/2$.] [10]

3A

(a) State the definition of the fundamental solution of Laplace's equation $\nabla^2 u = 0$ in two dimensions. Show that it is given by

$$G(\mathbf{r}, \mathbf{r}') = A \ln |\mathbf{r} - \mathbf{r}'| + B,$$

where A is a constant that you should specify, and B is an arbitrary constant. [5]

(b) Define the Green's function for Laplace's equation with Dirichlet boundary conditions in a two-dimensional region D with boundary C. Using the method of images, give an expression for the Green's function when D is the unit disc $0 \leqslant r < 1$. [You should verify that your solution satisfies the appropriate boundary conditions.] [5]

(c) Using Green's identity deduce that the solution to $\nabla^2 u = 0$ in the unit disc subject to the boundary condition $u = f$ on $r = 1$ is

$$u(r, \phi) = \frac{1}{2\pi} \int_0^{2\pi} f(\phi') P(r, \phi - \phi') d\phi'$$

for some function $P(r, \phi - \phi')$ that you should specify. [5]

(d) Assuming that the function f satisfies $f(\phi') \geqslant 0$ for all ϕ' in the range $0 \leqslant \phi' < 2\pi$, show that $u(r, \phi) \leqslant \frac{1+r}{1-r} u(0)$ for all $0 < r < 1$, where $u(0)$ is the value of u at the origin. [5]

4A

(a) Define the *residue* of a complex function $f(z)$ at a pole $z = z_0$, assuming f is analytic in a region $0 < |z - z_0| < R$ around z_0, where R is a positive real number. [2]

(b) Assuming $g(z)$ is analytic everywhere in the complex plane, what is the residue of $\frac{g(z)}{z - z_0}$ at $z = z_0$? [2]

(c) State the residue theorem, clearly including all the assumptions. Use it to show that $\frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2} dz = f'(z_0)$, where C is a contour encircling z_0 anticlockwise and f is analytic in the region enclosed by C. [6]

(d) Using contour integration methods compute the following integrals: [10]

(i) $$\int_{-\infty}^{+\infty} \frac{1}{(x^2 + 1)(x^2 + 4)} dx,$$

(ii) $$\int_0^{2\pi} \frac{d\theta}{5 - 3 \sin \theta}.$$

5A

(a) Using contour integration methods, show that for any complex number a

$$\int_{-\infty}^{\infty} e^{-(u+a)^2} du = \sqrt{\pi}.$$

[*Hint: You may use the fact that* $\int_{-\infty}^{+\infty} e^{-u^2} du = \sqrt{\pi}$.] [6]

(b) The Fourier transform of a function $f(x)$ that satisfies $f(x) \to 0$ as $|x| \to \infty$ is defined as

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx.$$

Consider a function $u(x,t)$ that satisfies

$$\frac{\partial u}{\partial t} = \lambda \frac{\partial^2 u}{\partial x^2}.$$

Using Fourier transform methods, assuming that $u(x,t) \to 0$ as $|x| \to \infty$, show that the solution is given by [8]

$$u(x,t) = \frac{1}{\sqrt{4\pi\lambda t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\lambda t}} u(y,0) dy.$$

(c) When $u(y,0) = \frac{\alpha}{\sqrt{\pi}} e^{-\alpha^2 y^2}$ with $\alpha > 0$, show that $u(x,t) = \frac{\beta}{\sqrt{\pi}} e^{-\beta^2 x^2}$, for a β that you should specify in terms of α, λ and t. What happens to $u(y,0)$ and $u(x,t)$ when $\alpha \to +\infty$? [6]

6B

(a) Explain what is meant by an *order n Cartesian tensor* and what it means for such a tensor to be *isotropic*.

Show that the only isotropic tensors of order 2 are of the form $a\delta_{ij}$, where a is a scalar. State (without proof) the most general form of an isotropic tensor of order 3. [9]

(b) Let $\mathbf{n_1}, \mathbf{n_2}, \mathbf{n_3}, \mathbf{n_4}$ be unit vectors in $\mathbb{R}^3$. Let V be the volume between two spheres with centres at the origin and radii R_1 and R_2, where $R_1 < R_2$.

Justifying your answers carefully, obtain expressions for:

(i) $\int_V (\mathbf{x} \cdot \mathbf{n_1})(\mathbf{x} \cdot \mathbf{n_2}) d^3x$,

(ii) $\int_V (\mathbf{x} \cdot \mathbf{n_1})(\mathbf{x} \cdot \mathbf{n_2})(\mathbf{x} \cdot \mathbf{n_3}) d^3x$,

(iii) $\int_V (\mathbf{x} \cdot \mathbf{n_1})(\mathbf{x} \cdot \mathbf{n_2})(\mathbf{x} \cdot \mathbf{n_3})(\mathbf{x} \cdot \mathbf{n_4}) d^3x$.

[*Hint: You may assume that the only isotropic tensors of order* 4 *are of the form* $c\delta_{ij}\delta_{kl} + d\delta_{ik}\delta_{jl} + e\delta_{il}\delta_{jk}$, *where* c, d *and* e *are scalars.*] [11]

7B

(a) A system has n degrees of freedom and undergoes small oscillations about an equilibrium point. Write down the general form of its Lagrangian in generalised coordinates, explaining any approximations used. Give the equations of motion, and explain what is meant by a *normal mode*, a *normal frequency* and a *zero mode* of the system. [4]

(b) Three beads of mass m, m and $2m$ are joined pairwise by identical ideal springs of spring constant k, with the masses and springs constrained to lie on a frictionless circular hoop of radius 1. The hoop is fixed and lies in a horizontal plane. Obtain the kinetic and potential energies of the system in terms of suitable generalized coordinates. Obtain the normal modes and normal frequencies. Briefly describe each normal mode physically. [9]

(c) At time $t = 0$ the masses are equally spaced around the hoop. The beads of mass m are at rest, and the bead of mass $2m$ is given a small initial angular velocity Ω. Obtain an equation for the state of the system at later times $t > 0$. [7]

8B

(a) Give the axioms for a group G. Explain what is meant by saying that H is a *subgroup* of G and that $\{g_1, \ldots, g_n\}$ is a *set of generators* for G. Explain what is meant by the *order* of G and the *order of an element* g of G, and state Lagrange's theorem. Explain what is meant by a *cyclic group* and an *isomorphism* between groups. [6]

(b) Now suppose G is a group of order 8. What are the possible orders of its elements? Show that if G has no elements of order greater than 2 then it is a direct product of cyclic groups. *[Hint: The direct product of groups $G_1, G_2, \ldots, G_n$ is the group with elements $(g_1, g_2, \ldots, g_n)$ (where $g_i \in G_i$), with identity $(I_{G_1}, \ldots, I_{G_n})$, where I_{G_i} is the identity of G_i. The group multiplication is defined by $(g_1, \ldots, g_n)(g_1', \ldots, g_n') = (g_1 g_1', \ldots, g_n g_n')$ and the inverse by $(g_1, \ldots, g_n)^{-1} = (g_1^{-1}, \ldots, g_n^{-1})$.]* [5]

(c) Now suppose that G has order 8 and is generated by elements a and b which have orders 4 and 2 respectively. Show that either (i) $ab = ba$ or (ii) $ab = ba^3$. Show that in case (i) G is uniquely defined up to isomorphism. Show that in case (ii) G is also uniquely defined up to isomorphism, and give a geometric representation of G and the generators a and b in this case. [7]

(d) Is the group identified in part (c)(ii) the only non-abelian group of order 8? Justify your answer carefully. [2]

9B

(a) Define a *homomorphism* between groups G and G'. Define what is meant by the *kernel* of a homomorphism and by a *normal subgroup* H of a group G. [4]

(b) Consider the set of matrices of the form

$$\mathsf{M} = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix},$$

where a and b are complex and $a \neq 0$. Show that this set forms a group G under matrix multiplication. Let G' be the subset of G such that $b = 0$. Show that G' is a subgroup of G and that the map

$$f : \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$$

defines a homomorphism. [8]

(c) Hence, or otherwise, identify a non-trivial normal subgroup of G and show that it is a normal subgroup. [8]

10B

(a) Define the *permutation group* S_n of n elements. Explain what is meant by an *m-dimensional representation* of a group G, by an *irreducible* representation of G, and by *inequivalent* representations of G. [4]

(b) Describe the conjugacy classes of S_4, stating the number of elements in each. [You need not prove these are the conjugacy classes.] [4]

(c) Describe two inequivalent one-dimensional representations of S_4, showing that they are representations. What are their kernels? Are the kernels subgroups? Justify your answers. What are the characters of the two representations? Verify that they are orthogonal. [6]

(d) Obtain the dimensions of the inequivalent irreducible finite-dimensional representations of S_4, justifying your answer carefully. [You may quote without proof any relevant theorems, provided they are clearly stated.] [6]

END OF PAPER