UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB

Tuesday, 28 May, 2019 9:00 am to 12:00 pm NST1

MATHEMATICS (1)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, ***6C****).*

Answers must be tied up in ***separate*** *bundles, marked* ***A, B or C*** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet ***must*** *be completed and attached to the bundle.*

A ***separate*** *green master cover sheet listing all the questions attempted* ***must*** *also be completed.*

Every cover sheet must bear your examination number and desk number.

Calculators are not permitted in this examination.

STATIONERY REQUIREMENTS
3 blue cover sheets and treasury tags
Green master cover sheet
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1C

(a) For vector fields $\mathbf{A}$ and $\mathbf{B}$ in three dimensions, show that

$$\boldsymbol{\nabla} \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{A} - \mathbf{B}(\boldsymbol{\nabla} \cdot \mathbf{A}) - (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B} + \mathbf{A}(\boldsymbol{\nabla} \cdot \mathbf{B})\,.$$ [3]

(b) State *Stokes's theorem*, taking care to define all the quantities which appear. [2]

(c) Elliptic cylindrical coordinates (u, v, z) are related to Cartesian coordinates (x, y, z) by

$$\begin{aligned} x &= a \cosh u \cos v\,, \\ y &= a \sinh u \sin v\,, \\ z &= z\,, \end{aligned}$$

where $u \geqslant 0$, $0 \leqslant v < 2\pi$, $-\infty < z < \infty$, and a is a positive real constant. Find the basis vectors $\mathbf{h}_u$, $\mathbf{h}_v$ and $\mathbf{h}_z$ defined by $d\mathbf{r} = \mathbf{h}_u\, du + \mathbf{h}_v\, dv + \mathbf{h}_z\, dz$, show that the coordinates are orthogonal, and find the scale factors h_u, h_v and h_z. [5]

(d) Describe the surfaces of constant u, the surfaces of constant v and the surfaces of constant z. [3]

(e) Consider the surface S with $z = c$ and

$$\frac{x^2}{\cosh^2 1} + \frac{y^2}{\sinh^2 1} \leqslant a^2\,,$$

where c is a positive constant and the normal to S points in the positive z direction. Calculate

$$\int_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{S}\,,$$

where $\mathbf{F} = (2 \sinh u \sin v, -2 \cosh u \cos v, \cosh u)$ in Cartesian coordinates. [7]

2C

The temperature, $T(x, y, t)$, in a two-dimensional bar satisfies

$$\frac{1}{\lambda}\frac{\partial T}{\partial t} = \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) ,$$

where $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$ and λ is a positive constant. The sides $x = 0$ and $x = a$ are held at fixed temperature $T = 0$, whereas the sides $y = 0$ and $y = b$ are insulating, i.e. $\left.\frac{\partial T}{\partial y}\right|_{y=0} = \left.\frac{\partial T}{\partial y}\right|_{y=b} = 0$.

(a) Using separation of variables and carefully explaining your working, show that the general solution can be written as

$$T(x, y, t) = \sum_{n,m} A_{nm} \sin\left(\frac{n\pi x}{a}\right) \cos\left(\frac{m\pi y}{b}\right) \exp\left[-\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right)\pi^2 \lambda\, t\right] ,$$

where A_{nm} are constants and you should specify the ranges of n and m in the sum. [8]

(b) The initial temperature is $T(x, y, 0) = x(a - x)\sin^2\left(\frac{2\pi y}{b}\right)$. What is $T(x, y, t)$? [10]

(c) What is the leading term in $T(x, y, t)$ for large t? [2]

3C

(a) Consider an inhomogeneous ordinary differential equation of the form

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x)\,, \qquad (*)$$

for $a \leqslant x \leqslant b$ subject to homogeneous boundary conditions at $x = a$ and b. Suppose that $y_a(x)$ and $y_b(x)$ are linearly independent solutions of the homogeneous equation (where $f(x) = 0$) and satisfy the boundary conditions at $x = a$ and $x = b$ respectively. Show that the Green's function can be written as

$$G(x,z) = \begin{cases} \frac{y_a(x)y_b(z)}{W(z)} & a \leqslant x \leqslant z\,, \\ \frac{y_b(x)y_a(z)}{W(z)} & z \leqslant x \leqslant b\,, \end{cases}$$

where $W(z) = y_a(z)y_b'(z) - y_b(z)y_a'(z)$. [5]

(b) Write an expression for $y(x)$, the solution of $(*)$, in terms of an integral involving f and G. [1]

(c) Find the general solution $y(x)$ of

$$\frac{d^2y}{dx^2} - \frac{3}{x}\frac{dy}{dx} + 3\frac{y}{x^2} = 0\,.$$

[*Hint: Consider* $y = x^n$.] [3]

(d) Consider the equation

$$\frac{d^2y}{dx^2} - \frac{3}{x}\frac{dy}{dx} + 3\frac{y}{x^2} = f(x)\,,$$

for $0 \leqslant x \leqslant 1$, with boundary conditions $y(0) = y(1) = 0$.

(i) Find the Green's function, $G(x,z)$. [3]

(ii) Find $y(x)$ when

$$f(x) = \begin{cases} 0 & 0 \leqslant x < \frac{1}{2} \\ x^2 & \frac{1}{2} \leqslant x \leqslant 1\,. \end{cases}$$ [8]

4C

The Fourier transform of a function $f(x)$ is given by

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx\,.$$

(a) Write down the corresponding expression for the inverse Fourier transform. [1]

(b) The convolution of two functions $f(x)$ and $g(x)$ is

$$h(x) = \int_{-\infty}^{\infty} f(z) g(x-z) dz\,.$$

Prove that $\tilde{h}(k) = \tilde{f}(k)\tilde{g}(k)$. [4]

(c) Find an expression for the Fourier transform of $x^n f(x)$ in terms of derivatives of $\tilde{f}(k)$. [4]

(d) Find the Fourier transform of the even function $q(x)$, where

$$q(x) = \begin{cases} 1 - x & 0 \leqslant x \leqslant 1 \\ 0 & x > 1\,. \end{cases}$$ [5]

(e) Find the Fourier transform of $p(x) = \int_{-1}^{1} q(x-z) dz$, where $q(x)$ is as defined in part (d). [6]

5A

(a) State the definition of the *adjoint* $\mathsf{A}^\dagger$ of a linear operator A with respect to a general inner product $\langle \mathbf{x} | \mathbf{y} \rangle$. In the special case of the standard dot product on complex vectors, give an expression for the adjoint operator. [4]

(b) State the definition of an *invertible* matrix. Assuming that the matrix A is diagonalizable, prove that A is invertible if and only if $\det(\mathsf{A})$ is nonzero. [5]

(c) Let M be an $n \times n$ matrix with real entries. Show that $\mathsf{M}^\mathrm{T}\mathsf{M}$ is real symmetric and that all its eigenvalues are non-negative. [5]

(d) Let B be a diagonalizable matrix such that $\mathsf{B}^k = \mathsf{0}$ for some integer k. Show that $\mathsf{B} = \mathsf{0}$. Give an example of a 2×2 non-zero matrix C such that $\mathsf{C}^2 = \mathsf{0}$. [6]

6C

(a) Let H be an $n \times n$ Hermitian matrix. Explain how to diagonalise H using an appropriate unitary matrix U to obtain a diagonal matrix Λ. What are the entries of Λ? [4]

(b) Explain how a quadratic form $\sum_{ij} A_{ij} x_i x_j$, where A_{ij} are real and $A_{ij} = A_{ji}$, can be written in the form $\sum_i a_i x'_i x'_i$. [3]

(c) Find the eigenvalues and eigenvectors of the matrix

$$\mathsf{B} = \begin{pmatrix} 1+c & 0 & 5-c \\ 0 & 3 & 0 \\ 5-c & 0 & 1+c \end{pmatrix} ,$$

where c is a real constant. [7]

(d) Describe the surface $\mathsf{x}^T \mathsf{B} \mathsf{x} = 1$, specifying the principal axes where appropriate. [*Hint: The type of surface may depend on the value of c.*] [6]

7C

(a) State the Cauchy-Riemann equations for an analytic function of $z = x + iy$, $f(z) = u(x, y) + iv(x, y)$, where x, y, u and v are real. [2]

(b) Show that curves of constant u and curves of constant v intersect at right angles. [3]

(c) Find the most general analytic function $f(z)$ with real part

$$u = e^{-x} \left[\left(x^2 - y^2\right) \cos y + 2xy \sin y \right] ,$$

writing your final answer in terms of z. [7]

(d) Find and classify the singularities and zeroes of the following functions (including any at the point at infinity)

$$\text{(i)} \quad \frac{z-4}{z^2 + iz + 6} , \qquad \text{(ii)} \quad \frac{e^{2z}}{\sinh z} .$$ [4]

(e) Find the power series expansion of

$$g(z) = \frac{1}{z - 2i}$$

about $z = 3$. Find the radius of convergence and comment. [4]

8C

(a) Define an *ordinary point* and a *regular singular point* for a second-order ordinary differential equation of the form

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0\,. \qquad [2]$$

(b) Classify the points $x = 0$ and $x = 1$ of

$$(1 - x^3)y''(x) - 6x^2y'(x) - 6xy(x) = 0\,.$$

Find a series solution about $x = 0$ subject to the boundary conditions $y(0) = 1$ and $y'(0) = 0$. Express the solution in closed form. [8]

(c) Find two linearly-independent series solutions about $x = 0$ of

$$4xy''(x) + 2(1 - x)y'(x) - y(x) = 0\,.$$

In particular, you should find the indicial equation, the recurrence relation and the radius of convergence. Express one solution in closed form. [10]

9B

(a) Explain what is meant by *Fermat's principle* and the *Euler-Lagrange equation.* [2]

(b) Using Fermat's principle, show that:

(i) when light is incident on a plane mirror the angle of incidence equals the angle of reflection;

(ii) if light crosses a planar boundary from a medium of refractive index μ_1 to a medium of refractive index μ_2, then

$$\sin(\theta_1)\mu_1 = \sin(\theta_2)\mu_2\,,$$

where θ_1 is the angle of incidence and θ_2 the angle of refraction. [8]

(c) A thin transparent medium lies in the semi-plane $-\infty < x < \infty$, $0 < y < \infty$. Its refractive index at the point (x, y) is given by $4\sqrt{y}$. A light ray travels from a source at $(-1, \frac{5}{4})$ to an observer at $(1, \frac{5}{4})$. Show that it may follow either of two possible paths, and derive the equations for these paths. [10]

10B (a) Consider a Sturm-Liouville operator of the form

$$\mathcal{L} = -\frac{d}{dx}\Big(\rho(x)\frac{d}{dx}\Big) + \sigma(x)\,.$$

The functionals $F[y]$ and $G[y]$ of real functions $y(x)$ are defined by

$$F[y] = \int_{-\infty}^{\infty} y(x)\mathcal{L}y(x)dx\,, \qquad G[y] = \int_{-\infty}^{\infty} w(x)\Big(y(x)\Big)^2 dx\,.$$

Assuming that $y(x) \to 0$ as $x \to \pm\infty$, show that the ratio $\Lambda[y] = F[y]/G[y]$ is extremized by solutions of the Sturm-Liouville eigenvalue problem

$$\mathcal{L}y(x) = \lambda w(x)y(x)\,.$$

What are the extremal values of $\Lambda[y]$? [7]

(b) A perturbed quantum harmonic oscillator is defined so that the expectation value of the energy of a particle is

$$E[\psi] = \int_{-\infty}^{\infty} \Big((\psi')^2 + (x^2 + \epsilon x^4)\psi^2\Big)dx$$

when its state is defined by a real wave function $\psi(x)$ obeying

$$\int_{-\infty}^{\infty} \Big(\psi(x)\Big)^2 dx = 1$$

and $\psi(x) \to 0$ as $x \to \pm\infty$.

Consider the case $\epsilon = 2$, and obtain the minimum expectation value of the energy for a particle wave function of the form $\psi_{\text{trial}}(x) = C\exp(-\frac{\alpha}{2}x^2)$, where C and α are real and $\alpha > 0$. Define the relevant Sturm-Liouville eigenvalue problem. Explain why the calculated minimum expectation value gives an upper bound on the smallest eigenvalue for this problem.

[Hint: You may use the result that, for $\alpha > 0$,

$$\int_{-\infty}^{\infty} x^{2n}\exp(-\alpha x^2)dx = \frac{(2n)!}{2^{2n}n!}\sqrt{\frac{\pi}{\alpha^{2n+1}}}\ .]$$

[8]

(c) Without carrying out an explicit calculation, explain how you might improve this bound. [2]

(d) Is there a minimum energy if $\epsilon < 0$? Justify your answer briefly. [3]

END OF PAPER