MATHEMATICS (2)

Before you begin read these instructions carefully:

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

You may submit answers to all of section A, and to no more than five questions from section B.

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on one side of the paper only and begin each answer on a separate sheet. (For this purpose, your section A attempts should be considered as one single answer.)

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Tie up all of your section A answer in a single bundle, with a completed blue cover sheet.

Each section B question has a number and a letter (for example, 11X). Answers to these questions must be tied up in separate bundles, marked R, S, T, V, W, X, Y or Z according to the letter affixed to each question. Do not join the bundles together. For each bundle, a blue cover sheet must be completed and attached to the bundle, with the correct letter R, S, T, V, W, X, Y or Z written in the section box.

A separate green master cover sheet listing all the questions attempted must also be completed. (Your section A answer may be recorded just as A: there is no need to list each individual short question.)

Every cover sheet must bear your examination number and desk number.

Calculators are not permitted in this examination.

STATIONERY REQUIREMENTS

6 blue cover sheets and treasury tags
Green master cover sheet
Single-sided script paper
Rough paper

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.
1

For \( z \in \mathbb{C} \), express

\[
f = \frac{z + 1}{z^2 + 1}
\]

as the sum of two partial fractions where the denominator of each is a linear function of \( z \). \[2\]

2

Determine the general term in the power series in \( x \) expanded around \( x = 0 \) for:

(a) \( \exp(x^3) \); \[1\]

(b) \( \int_0^x \exp(y^3) \, dy \). \[1\]

3

Find the general solution of

\[
\frac{dy}{dt} + 4\frac{y}{t} = 3.
\] \[2\]

4

Consider the integral \( I_n = \int (\ln x)^n \, dx \) for \( x > 0 \) and integer \( n \geq 0 \).

(a) Express \( I_{n+1} \) in terms of \( I_n \). \[1\]

(b) Evaluate \( I_1 \). \[1\]

5

Consider the function \( f(x, y) = x^2 - x + xy - 3y - y^2 - 1 \).

(a) Find the stationary point of \( f(x, y) \). \[1\]

(b) Classify the stationary point. \[1\]
6

(a) Express \( P(A \cup B \cup C) \) in terms of the probabilities of the individual events \( A, B, C \) and the intersections between these events. [1]

(b) If \( D \subset (A \cup B \cup C) \), determine \( P((A \cup B \cup C)|D) \). [1]

7

Determine the value(s) of \( \lambda \) for which there are non-trivial solutions for \( x \) to

\[
\begin{pmatrix}
1 & 2 \\
2 & 1
\end{pmatrix} x = \lambda \begin{pmatrix}
0 & 2 \\
2 & 0
\end{pmatrix} x.
\]

[2]

8

Determine the coefficients \( a_n \) and \( b_n \) for the Fourier Series

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))
\]

of \( f(x) = \cos^3 x \sin x \) in the interval \(-\pi < x \leq \pi\). \[Hint: \cos^3 x \sin x = (\cos^2 x)(\cos x \sin x).\] [2]

9

A solid sphere of radius \( a \) has a density distribution \( \rho(r) = 1 + r \), where \( r \) is the distance from the centre of the sphere. What is the mass of the sphere? [2]

10

(a) Determine the vector area \( \mathbf{S} \) (in Cartesian coordinates) of the outside of the shell defined in spherical polar coordinates \((r, \theta, \phi)\) by \( r = a \), \( 0 \leq \theta \leq \pi/4 \), where \( \theta = 0 \) is in the direction of the \( z \)-axis. [1]

(b) What is the projection of \( \mathbf{S} \) in the direction given by the Cartesian vector \( \mathbf{q}^T = (1, 1, 1) \)? [1]
SECTION B

11X

(a) (i) Draw right-handed Cartesian axes and label the following points: the origin $O (0, 0, 0)$; $A (1, 0, 0)$; $B (0, 1, 0)$; $C (0, 0, 1)$. [2]

(ii) Draw position vectors $e = (0, \frac{1}{2}, \frac{1}{2})$, $f = (\frac{1}{2}, 0, \frac{1}{2})$ and $g = (\frac{1}{2}, \frac{1}{2}, 0)$, leading to the points $E$, $F$ and $G$, respectively. Calculate the volume of the parallelepiped defined by $e$, $f$ and $g$. Explain briefly why the result of this calculation is relevant to the question of whether $\{e, f, g\}$ can be used as a basis. [6]

(iii) Write the vector $\overrightarrow{OA}$ in terms of the basis $\{e, f, g\}$. [2]

(b) Calculate $\hat{n}$, the unit vector normal to the plane containing the points $E$, $F$ and $G$. Write down a vector equation for this plane and hence find its Cartesian equation. Calculate the perpendicular distance from the origin to the plane. [6]

(c) The point $R$ with position vector $r$ lies on the line $EF$. Show that

$$r \times (f - e) = e \times f.$$ 

Find a formula for the shortest distance from the point $C$ to the line $EF$. Calculate this distance. [The vector product can be written as either $e \times f$ or $e \wedge f$.] [4]

12R

(a) Find the stationary points and stationary values of the function

$$f(x, y) = \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3}y^3.$$ [6]

(b) Determine the nature of the stationary points. [4]

(c) Sketch the contours of the function in the range $|x| \leq 2$, $|y| \leq 2$. [6]

(d) Add arrows to your sketch, showing the direction of the gradient vector $\nabla f$, to highlight the behaviour of the function. [4]
Consider the vector field
\[ \mathbf{F}(\mathbf{x}) = \begin{pmatrix}
2\alpha x + \alpha^2 y + \alpha z \\
\alpha^2 x + \beta z \\
\alpha x + \beta y^2
\end{pmatrix}, \]
where \(\alpha, \beta\) are real parameters.

(a) Evaluate \(\int \mathbf{F} \cdot d\mathbf{x}\) along the straight line from \((0,0,0)\) to \((1,1,1)\). [4]

(b) Evaluate \(\int \mathbf{F} \cdot d\mathbf{x}\) along the path \(\mathbf{x}(t) = (t, t^2, t^3)\), with \(0 \leq t \leq 1\). [6]

(c) Determine the value of \(\beta\) for which \(\mathbf{F}\) is a conservative field. [3]

(d) For the value of \(\beta\) determined in part (c), find the scalar field \(\Phi(\mathbf{x})\) such that \(\mathbf{F} = \nabla \Phi\) and \(\Phi(\mathbf{x}) = 0\) at the origin. Determine the value(s) of \(\alpha\) such that \(\Phi(\mathbf{x}) = 1\) at \((1,1,1)\). [7]
Let $X$ and $Y$ be continuous independent random variables distributed according to probability density functions $f(x)$ and $g(y)$, respectively.

(a) Give an expression for the probability $P(a \leq X \leq b)$, where $a$ and $b$ are real parameters and $a < b$. [2]

(b) Consider the new random variable $U = X + c$, with $c$ being a real constant.
   (i) Find $P(a \leq U \leq b)$. [2]
   (ii) By considering $P(u \leq U \leq u + du)$, find $t(u)$, the probability density function for $U$. [3]

(c) Consider the random variable $Z = X + Y$.
   (i) For a given value of $Y$, find $P(z \leq Z \leq z + dz | Y = y)$. [2]
   (ii) Show that the probability density function of $Z$, $h(z)$, is given by
       $$h(z) = \int_{-\infty}^{\infty} g(y) f(z - y) dy.$$ [3]

(d) Let $f(x) = \pi^{-1} (1 + x^2)^{-1}$ for $-\infty < x < \infty$ and $Y$ be uniformly distributed between $-1/2$ and $1/2$.
   (i) Find the probability density function $g(y)$. [1]
   (ii) Find the probability density function $h(z)$ for the random variable $Z$ defined in (c). [2]
   (iii) Sketch $h(z)$ and find the most probable and mean values of $Z$. [5]
15Y

(a) Solve the following ordinary differential equations:

(i) \[
\frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 13y = 0, \quad \text{with} \quad y(0) = \pi \quad \text{and} \quad y(-\pi/2) = 1; \quad [3]
\]

(ii) \[
\frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} + y = 2xe^{-x}. \quad [6]
\]

(b) Consider the pair of ordinary differential equations:

\[
\frac{du}{dt} = -3u + v, \quad \frac{dv}{dt} = -5u + v. \]

(i) Rewrite these equations to obtain a second-order differential equation for \( u(t) \).

(ii) Determine the general solution for \( u(t) \).

(iii) Find the solution for \( u(t) \) satisfying \( u(0) = 1 \) and \( v(0) = 1 \).

16V

(a) Evaluate \[
\int \int_S (z + y^3) \, dS,
\]

where \( S \) is the total surface made from the vertical cylinder \( x^2 + y^2 = a^2 \) with \( 0 \leq z \leq b \), the flat disc \( x^2 + y^2 \leq a^2 \) in the \( z = b \) plane \( (b > a > 0) \), and the hemispherical indentation \( x^2 + y^2 + z^2 = a^2 \) with \( z \geq 0 \). \[10\]

(b) Calculate the flux of the vector field \[
\mathbf{F} = e^{-x} \hat{i} + \frac{1}{x} \left( \frac{1}{(\ln x)^2 + 1} \right) \hat{j} + z \hat{k}
\]

through each of the six faces of the axes-aligned unit cube with two vertices at \( (0,0,0) \) and \( (1,1,1) \). Hence determine the total flux out of the cube. \[10\]
(a) Consider the equation $x^T M = \lambda x^T$, where $x^T = (x, y, z)$ and 

$$M = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

Determine the values of $\lambda$ for which there are non-trivial solutions and determine the corresponding solutions. [4]

(b) The elements of the non-singular matrix $A$ are given by $a_{ij}$ and those of matrix $B$ by $b_{ij}$. Both matrices are $n \times n$, where $n > 1$. Write down the following determinants in terms of $|A|$ and $|B|$: 

(i) $|AB|$; [2]  

(ii) $|\alpha A^{-1}|$, for constant $\alpha$; [2]  

(iii) $|C|$, where the elements of $C$ are related to those of $A$ through $c_{ij} = \beta_i \gamma_j a_{ij}$ and $\beta_i, \gamma_j$ are a set of constants. [3]

(c) The $n \times n$ matrix $Q$ (with $n > 1$) has eigenvalues $\lambda_i$ with corresponding eigenvectors $e_i$. From these we define two additional matrices, $D$ and $E$. The matrix $D$ is diagonal with non-zero elements $d_{ii} = \lambda_i$. The columns of the matrix $E$ are the eigenvectors (i.e. $E = (e_1, e_2, \ldots, e_n)$). You may assume $|E| \neq 0$.

(i) Express $Q$ in terms of $D$ and $E$. [4]  

(ii) The $m \times n$ matrix $V_k$ is determined recursively by $V_k = V_{k-1}Q$, for $k \geq 1$. Express $V_k$ in terms of $V_0, D$ and $E$. [5]
18S

Consider the periodic functions \( f(x) \) and \( g(x) \), both with period 2, defined over the range \(-1 < x \leq 1\) as \( f(x) = \cosh x \) and \( g(x) = \sinh x \).

(a) Determine the Fourier Series representation of \( f(x) \). Hence or otherwise, show that

\[
f(1) = \cosh(1) = \sinh(1) \left( 1 + p \sum_{n=1}^{\infty} \frac{1}{1 + n^2 q^2} \right),
\]

and determine the constants \( p \) and \( q \). [8]

(b) Determine the Fourier Series representation of \( g(x) \). [Hint: This can be achieved without computing any further integrals.] [4]

(c) State Parseval’s theorem and use it to show that

\[
\int_{-1}^{1} (f(x) - g(x))^2 \, dx = \sinh(2).
\] [8]
(a) Use Lagrange multipliers to determine the height and radius of the circular cylinder (with ends normal to the axis of the cylinder) of volume $V$ that has the minimal total surface area $S$ and calculate this area. 

(b) Consider a cone of height $h$ and base radius $r$. The axis of the cone is aligned with the $z$-axis and the base is normal to the $z$-axis. The apex of the cone, located at $(0,0,R)$ with constant $R$, lies on the surface of a sphere that is centred on the origin $O$. Using a Lagrange multiplier $\lambda$ to enforce the condition that the cone is inscribed by the sphere (see figure), determine the values of $r$ and $h$ that maximise the volume of the cone. Calculate the maximum volume of the cone as a fraction of the volume of the sphere. [Hint: the volume of the cone is given by $V = \frac{1}{3}\pi r^2 h$.] 

(c) Find the minimum of the function 

$$f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i, \quad \text{with} \quad x_i > 0 \quad \text{and} \quad i = 1, \ldots, n,$$

subject to constraint 

$$\prod_{i=1}^{n} x_i = a, \quad \text{with constant} \quad a \in \mathbb{R}, \quad a > 0.$$

Hence or otherwise, deduce an inequality relating the arithmetic mean $\frac{1}{n} \sum_{i=1}^{n} x_i$ and geometric mean $(\prod_{i=1}^{n} x_i)^{1/n}$. 

Figure: Sketch of a cone of height $h$ and base radius $r$ inscribed into a sphere centred on $O$. 

[8]
(a) By using the method of separation of variables, find a non-trivial solution for each of the following first-order partial differential equations for real-valued functions $u(x, y)$:

(i) \[ \frac{\partial u}{\partial x} - x \frac{\partial u}{\partial y} = 0 ; \tag{3} \]

(ii) \[ x \frac{\partial u}{\partial x} - 2y \frac{\partial u}{\partial y} = 0 . \tag{3} \]

(b) A function $T(x, t)$ obeys the diffusion equation

\[ \frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \tag{\dagger} \]

for a constant $\kappa > 0$, non-negative $t$ and $-\infty < x < \infty$. The initial condition is given by $T(x, t = 0) = T_0 \exp(-x^2/L^2)$, where $T_0$ and $L$ are positive constants.

Substitute a solution of the form $T(x, t) = F(t) \exp(-x^2H(t))$ into equation (\dagger), where $F(t)$ and $H(t)$ are positive differentiable functions, to obtain two simultaneous equations containing $F(t)$ and $H(t)$. Solve these equations for $F(t)$ and $H(t)$. Thus show that

\[ H(t) = 1/(4\kappa t + L^2) \quad \text{and} \quad F(t) = T_0 L/\sqrt{4\kappa t + L^2}. \tag{8} \]

Write out the solution of equation (\dagger).

END OF PAPER