UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Monday, 10 June, 2019 9:00 am to 12:00 pm NST0, CST0

MATHEMATICS (1)

Before you begin read these instructions carefully:

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

You may submit answers to ***all*** *of section A, and to no more than* ***five*** *questions from section B.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on one side of the paper only and begin each answer on a separate sheet. *(For this purpose, your section A attempts should be considered as one single answer.)*

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Tie up ***all of your section A answer*** *in a single bundle, with a completed blue cover sheet.*

Each section B question has a number and a letter (for example, ***11Z****). Answers to these questions must be tied up in* ***separate*** *bundles, marked* ***R, S, T, V, W, X, Y*** *or* ***Z*** *according to the letter affixed to each question.* ***Do not join the bundles together****. For each bundle, a blue cover sheet* ***must*** *be completed and attached to the bundle, with the correct letter* ***R, S, T, V, W, X, Y*** *or* ***Z*** *written in the section box.*

A ***separate*** *green master cover sheet listing all the questions attempted* ***must*** *also be completed. (Your section A answer may be recorded just as A: there is no need to list each individual short question.)*

Every cover sheet must bear your examination number and desk number.

Calculators are not permitted in this examination.

STATIONERY REQUIREMENTS

6 blue cover sheets and treasury tags
Green master cover sheet
Single-sided script paper
Rough paper

SPECIAL REQUIREMENTS

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1

Solve for x: $2x^2 + x \leqslant 6$. [2]

2

Find the coordinates of the centre of the ellipse: $x^2 - 4x + 4y^2 - 8y = -4$. [2]

3

(a) In the interval $0 \leqslant x \leqslant \pi/2$, for what value of the constant c is the straight line, $y = x + c$, a tangent to the curve $y = \sin 2x$? [1]

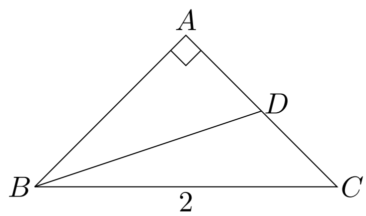
(b) What is the gradient of the normal to the curve at the point where it meets the straight line? [1]

4

The diagram below shows an isosceles triangle ABC with a right angle at A. D is the midpoint of AC. The length of BC is 2 units. Calculate:

(a) The length AD; [1]

(b) The cosine of the angle ADB. [1]



5

Solve for x in the interval $0 \leqslant x \leqslant \pi$: $\cos 2x = \sin x$. [2]

6

Calculate $\int_1^y x^2 \ln x \, dx$ for $y > 1$. [2]

7

Evaluate:

(a) $\displaystyle\sum_{n=2}^{101} 2n$; [1]

(b) $\displaystyle\sum_{n=1}^{5} 4^n$. [1]

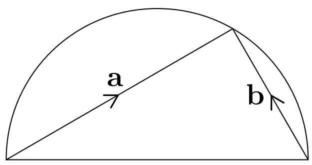
8

Find the constants a and b in:

$$\frac{x^2+1}{(x+1)(x+2)^2} = \frac{a}{x+1} + \frac{b}{x+2} - \frac{5}{(x+2)^2} \,.$$ [2]

9

(a) The diagram shows a semicircle with vectors **a** and **b** drawn from the two ends of the diameter to a point on the perimeter. Calculate $\mathbf{a} \cdot \mathbf{b}$. [1]



(b) How are $|\mathbf{a}|$ and $|\mathbf{b}|$ related if the area of the triangle is half the area of the semicircle? [1]

10

For the vectors $\mathbf{s} = (\sqrt{3}, 1)$ and $\mathbf{t} = (1, \sqrt{3})$, calculate:

(a) The magnitude of $\mathbf{s}$; [1]

(b) The vector $\mathbf{s} - 5\mathbf{t}$. [1]

SECTION B

11Z

(a) Use De Moivre's theorem to express $8\cos(6\theta) + 15\sin(4\theta)\sin(2\theta)$ in terms of $\cos\theta$ and $\sin\theta$. [4]

(b) For $z \in \mathbb{C}$, with $z = x + iy$, find the real and imaginary parts of $\tan(z^*)$. [5]

(c) Find the locus that solves $|2z - z^* - 3i| = 2$, and sketch it on an Argand diagram. [4]

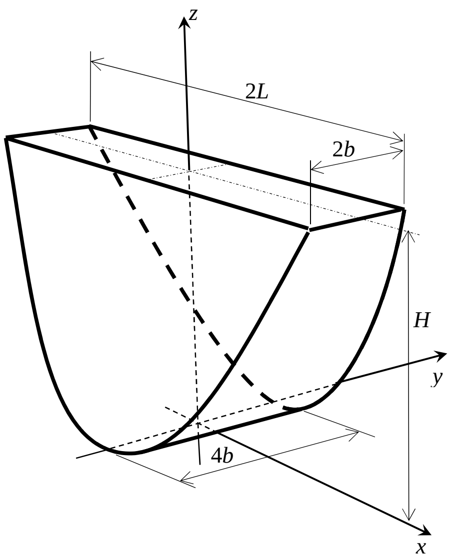
(d) By writing $z = re^{i\theta}$ (where $r, \theta \in \mathbb{R}$), find the real and imaginary parts of $f = \ln(z^{1+i})$ in terms of r and θ. By limiting the argument of z to range from $-\pi$ to π, sketch on an Argand Diagram the solution to $\text{Re}(f) = 0$. [7]

12S

A new rail link is planned. To support the tracks, the link requires the construction of an embankment across a small valley.

(a) The material for the embankment is stockpiled in a single circular heap of volume V and constant outer radius R. Within the heap the height profile is given by $R\exp(-r/R)$ with $r \leqslant R$. Here, r is the radial distance from the vertical axis of the heap. Determine R in terms of V. [5]

(b) The embankment (see figure) is defined by the region $|x| \leqslant L$, $|y| \leqslant b(2 - z/H)$ and $H(x/L)^2 \leqslant z \leqslant H$, where x is oriented across the valley and z upwards. Here, H is the height of the embankment, $2L$ its length and $2b$ its width at the top. Calculate the volume of the embankment. [9]



(c) Later, it is realised that there needs to be a tunnel through the embankment for a cycle path along the bottom of the valley. The tunnel is cylindrical with diameter D (where $D < L^2/H$ and $D < H$), and is centred on $x = 0$, $z = \frac{1}{2}D$ and has its axis in the y direction. Looking along the y-axis, sketch in the $x - z$ plane the embankment and the tunnel. Calculate the volume of the material that needs to be removed from the embankment to create the tunnel. [6]

13Y

(a) Determine whether or not the differential form

$$(2x + e^y)dx + (xe^y - \cos y)dy$$

is exact. Hence or otherwise, solve the equation

$$2x + e^y + (xe^y - \cos y)\frac{dy}{dx} = 0\,,$$

with $y(1) = \pi/2$. [5]

(b) Solve the differential equations:

(i) $\dfrac{1}{x}\dfrac{dy}{dx} + y - 5e^{-x^2} = 0\,;$ [4]

(ii) $\dfrac{dy}{dx} + (1 + \ln x)y = x^{-x}\,,$ with $y(1) = 2\,.$ [5]

(c) The Laplace equation $\nabla^2 u = 0$ can be written in spherical polar coordinates (r, θ, ϕ) as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0\,.$$

Show that

$$u = \left(r + \frac{1}{r^2}\right)\sin\theta\,\cos\phi$$

is a solution. [6]

14R

(a) The energy, $E(m, v)$, of a relativistic particle of rest mass m and speed v is

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}}\,,$$

where c, the speed of light, is a constant.

(i) Express dE/E, the small fractional change in energy, in terms of the small fractional changes dm/m and dv/v.

(ii) Two particles, A and B, moving at 90% and 91% the speed of light respectively, have equal energy. Find the difference between their rest masses, assuming it to be small, in terms of the rest mass of A. Hence identify the particle with the larger rest mass. [4]

(b) If $u(x, y) = \phi(xy) + \sqrt{xy}\;\psi(y/x)$, where ϕ and ψ are twice-differentiable functions of their arguments, show that

$$x^2\frac{\partial^2 u}{\partial x^2} - y^2\frac{\partial^2 u}{\partial y^2} = 0\,.$$ [10]

15V

(a) State Taylor's theorem by giving the series expansion about $x = a$ of a function $f(x)$ that is n times differentiable, showing the first n terms, together with an expression for the remainder term R_n. [4]

(b) Find the Taylor series expansion about $x = a$, up to and including the term proportional to $(x - a)^m$, of the following functions (you may use standard Taylor series expansions without proof):

(i) $\sin\dfrac{\pi e^x}{2}$ for $a = 0$ and $m = 2$; [4]

(ii) $\dfrac{\sinh(x+1)}{x+2}$ for $a = -1$ and $m = 4$; [6]

(iii) $\dfrac{\ln(1+x^3)}{\cosh(x)}$ for $a = 0$ and $m = 6$. [6]

16W

A box contains 2 blue and 3 non-blue but otherwise identical balls. An experiment consists of three consecutive events: drawing a ball from the box, returning or not returning it back to the box, and drawing a second ball from the box. For example, an experiment might consist of (i) the event B_1 of drawing a blue ball in a first draw, (ii) the event R of returning the ball to the box and (iii) the event $\overline{B}_2$ of drawing a non-blue ball in a second draw. The probability of event R is $P(R) = r$.

(a) Find the sample space of this experiment and the probabilities of all possible outcomes using notations such as $P(B_1 \cap \overline{R} \cap B_2)$ for the probability of outcome $B_1 \cap \overline{R} \cap B_2$. [8]

(b) Find the probability, $P(B_2)$, that the second ball is blue if:

(i) $r = 0$; [1]

(ii) $r = 1$; [1]

(iii) r is arbitrary in the range, $0 \leqslant r \leqslant 1$. [1]

(c) Find:

(i) $P(B_1 \cap B_2)$; [1]

(ii) $P(R|B_1 \cap B_2)$. [2]

(d) Consider the general case of a box containing $N_B > 1$ of blue and $N - N_B > 1$ of non-blue but otherwise identical balls. For the experiment described above, find $P(R|\overline{B}_1 \cap B_2)$ in this general case. By sketching this probability as function of r for fixed N and N_B, show that

$$P(R|\overline{B}_1 \cap B_2) \leqslant r\,.$$ [6]

17T

(a) Evaluate the following indefinite integrals for real x:

(i) $\displaystyle\int e^x \sinh 3x \, dx\,;$ [3]

(ii) $\displaystyle\int \frac{\arctan x}{x^2}\, dx\,.$ [4]

(b) Evaluate the definite integral

$$\int_{e^3}^{e^4} \frac{3\ln x - 4}{x\ln^2 x - 3x\ln x + 2x}\, dx\,.$$ [5]

(c) Express I_n in terms of I_{n-1}, for integer $n \geqslant 1$, where

$$I_n \;=\; \int_0^{\pi} x^{2n} \cos x \, dx\,,$$ [8]

and evaluate I_3.

18Z

(a) The real orthogonal matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & a \end{pmatrix},$$

has the property that $\mathbf{A}^2 = -\mathbf{I}$, where $\mathbf{I}$ is the identity matrix.

(i) Determine the possible matrices $\mathbf{A}$. [6]

(ii) For each possible $\mathbf{A}$, find a real orthogonal matrix $\mathbf{B}$ with trace($\mathbf{B}$) < 0 such that $\mathbf{B}^2 = \mathbf{A}$. [5]

(b) A study by students of $n \geqslant 2$ earthworms recorded that worm i, found a distance x_i from a river, had length y_i. The students notice a correlation between x_i and y_i and propose that the predicted length p_i of worm i is given by $p_i = \alpha + \beta x_i$. They write this for all the worms as $\mathbf{p} = \mathbf{Mr}$, where the components of the vector $\mathbf{p}$ are the predictions p_i with each row of $\mathbf{M}$ and $\mathbf{p}$ containing information about one of the worms. The vector $\mathbf{r} = (\alpha, \beta)^T$ contains the model parameters as components.

(i) Write down an expression for the $n \times 2$ matrix $\mathbf{M}$ in terms of x_i. [3]

(ii) The components of $\mathbf{r}$ are estimated from the solution to $\mathbf{M}^T\mathbf{Mr} = \mathbf{M}^T\mathbf{y}$, where the components of the vector $\mathbf{y}$ are the observed worm lengths y_i. Solve this system and give explicit expressions for α and β in terms of the quantities $\overline{x}$, $\overline{x^2}$, $\overline{y}$, $\overline{xy}$, where $\overline{f} = (\sum_{i=1}^{n} f_i)/n$. [6]

19V*

(a) Determine whether the following series converge or diverge:

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(2n-1)\pi}{n}\,;$ [4]

(ii) $\displaystyle\sum_{n=0}^{\infty}\sum_{m=n}^{\infty} \frac{1}{2^m}\,.$ [4]

(b) The real function $f(x)$ is defined as

$$f(x) = x^2 - 2\varepsilon x - 1\,,$$

where the parameter ε is small (i.e. $|\varepsilon| \ll 1$). Suppose x_i is the i-th Newton-Raphson iteration (with $x_0 = 1$) for the positive root x_* of $f(x)$ (i.e. $f(x_*) = 0$). By considering the leading order term in the Taylor series expansion, show that $|x_i - x_*| \propto \varepsilon^{n_i}$, where:

(i) $n_0 = 1\,;$ [2]

(ii) $n_1 = 2\,;$ [4]

(iii) $n_2 > 3\,.$ [6]

20R*

(a) The interval $a \leqslant x \leqslant b$ of the x-axis, with $0 < a < b$, is divided into n equal sub-intervals of width $\Delta x = (b-a)/n$. For each sub-interval k, $1 \leqslant k \leqslant n$, a rectangle of width Δx and height $y_k = (a + k\Delta x)^2$ is constructed. Find the sum of the areas of the rectangles as a function of n and show that, as $n \to \infty$, it tends to the area under the parabola $y = x^2$ between $x = a$ and $x = b$. [*Hint:* $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$.] [6]

(b) Differentiate the function

$$f(x) = \int_{\sin x}^{\cos x} \exp(-xt^4)\, dt$$

with respect to x. It is not necessary to evaluate integrals that arise. [6]

(c) Show that for positive parameters p and q,

$$I = \int_0^{\pi/2} \frac{dx}{p\cos^2 x + q\sin^2 x} = \frac{\pi}{2}\frac{1}{\sqrt{pq}}\,.$$ [4]

By considering $\partial I/\partial p$ and $\partial I/\partial q$, evaluate

$$J = \int_0^{\pi/2} \frac{dx}{(p\cos^2 x + q\sin^2 x)^2}\,.$$ [4]

END OF PAPER