UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB

Friday, 1 June, 2018 9:00 am to 12:00 pm

MATHEMATICS (2)

Before you begin read these instructions carefully:

*You may submit answers to no more than **six** questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right-hand margin.

*Write on **one** side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

*Each question has a number and a letter (for example, **6C**).*

*Answers must be tied up in **separate** bundles, marked **A, B or C** according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

*A **separate** green master cover sheet listing all the questions attempted **must** also be completed.*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
3 blue cover sheets and treasury tags
Green master cover sheet
Script paper

SPECIAL REQUIREMENTS
Calculator - students are permitted to bring an approved calculator.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1C

Consider the eigenvalue problem

$$-\left(1-x^2\right)y''+xy'=n^2y, \qquad -1\leqslant x\leqslant 1, \qquad (\star)$$

where $n\geqslant 0$ is an integer.

(a) Rewrite equation $(\star)$ in Sturm–Liouville form and determine the weight function $w(x)$. Show that any two eigenfunctions y_n and y_m of $(\star)$ with $n\neq m$ satisfy the orthogonality condition

$$\int_{-1}^{1} w(x)y_n(x)y_m(x)\ \mathrm{d}x=0\,,$$

provided the y_n and their derivatives are finite at $x=\pm1$. [5]

(b) The eigenfunctions y_n of $(\star)$ are n^{th}-order polynomials that satisfy $y_n(1)=1$. Calculate y_0, y_1 and y_2 explicitly. Also calculate I_0 and I_1, where

$$I_n=\int_{-1}^{1} wy_n^2\ \mathrm{d}x$$

is the weighted norm of y_n. [5]

(c) Consider now the equation for $Z(x)$,

$$\left(1-x^2\right)Z''-xZ'+\gamma^2Z=e^{\varepsilon x}\,, \qquad -1\leqslant x\leqslant 1, \qquad (\dagger)$$

where γ is a real non-integer constant and $\varepsilon\ll 1$ is a positive real constant.

(i) By looking for an expansion of $Z(x)$ in terms of the eigenfunctions y_n of $(\star)$, or otherwise, and expanding the right-hand side of $(\dagger)$ in powers of ε, find an expression for $Z(x)$ of the form

$$Z(x)=A+\varepsilon B+\varepsilon^2C+O(\varepsilon^3)\,.$$

You should write A, B and C in terms of γ, y_0, y_1 and y_2. You do not need to calculate any of the $O(\varepsilon^3)$ terms. [5]

(ii) Now suppose $\gamma^2=5$. Using your answers to part (b), or otherwise, show that

$$\int_{-1}^{1}\left[\left(1-x^2\right)^{-1/2}+\left(1-x^2\right)^{1/2}\right]Z(x)\,\mathrm{d}x=\frac{3\pi}{10}+\varepsilon^2\frac{\pi}{80}+O(\varepsilon^3)\,.$$

You may use without proof that $I_2=\pi/2$. [5]

2B

Consider Laplace's equation in plane polar coordinates

$$\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r}\frac{\partial \Psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Psi}{\partial \phi^2} = 0, \qquad (\star)$$

where $0 \leqslant \phi < 2\pi$ is a periodic coordinate, $\Psi(r, \phi)$ is single-valued and finite inside the disk of radius $R > 0$ centred at the origin.

(a) Use separation of variables to show that the general solution can be written as: [4]

$$\Psi(r, \phi) = A_0 + \sum_{n=1}^{\infty} r^n \Big[A_n \cos(n\phi) + B_n \sin(n\phi) \Big].$$

(b) Assume Ψ satisfies the boundary condition $\Psi(R, \phi) = f(\phi)$ for $0 \leqslant \phi < 2\pi$. Show that the value of Ψ at the centre of the disk is equal to the average value of f on the circle of radius R. [6]

(c) Compute the values of A_0, A_n and B_n when $R = 2$ and [6]

$$f(\phi) = \begin{cases} 1 & \text{if } 0 \leqslant \phi < \pi \\ \cos^2(\phi) & \text{if } \pi \leqslant \phi < 2\pi. \end{cases}$$

(d) Show that any solution Ψ of Laplace's equation $(\star)$ on the disk attains its maximum value on the boundary of the disk.

[*Hint: Use part (b) to show that the value of* Ψ *at any point in the interior of the disk is the average of* Ψ *on a circle surrounding that point.*] [4]

3B Consider Poisson's equation on a volume V in $\mathbb{R}^3$ with boundary conditions specified on the surface S:

$$\begin{cases} \nabla^2\Phi = \rho(\mathbf{r}) & \text{on } V \\ \Phi = f(\mathbf{r}) & \text{on } S. \end{cases}$$

(a) State the definition of a Green's function for Poisson's equation with the boundary conditions on the surface S as above. [4]

(b) Using Green's identity, show that the solution to Poisson's equation can be expressed as [4]

$$\Phi(\mathbf{r}') = \int_V \rho(\mathbf{r})G(\mathbf{r},\mathbf{r}')dV + \int_S f(\mathbf{r})\frac{\partial G}{\partial n}dS$$

where G is the Green's function.

(c) Write down the fundamental solution in $\mathbb{R}^3$. Hence, find the Green's function in the case where $V = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leqslant 1\}$ is the interior of the sphere of radius 1 centred at the origin. [6]

(d) Use the method of images to determine the Green's function when $V = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leqslant 1, z \geqslant 0\}$ is the interior of the half-sphere ($z \geqslant 0$). [6]

4A

(a) State Cauchy's theorem and Cauchy's formula, clearly stating the assumptions about the integration contour used. [4]

(b) The extension of Cauchy's formula is

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz,$$

where $f^{(n)}(z) = \frac{d^n f}{dz^n}$.

Use this formula to evaluate

$$\oint_C \frac{\sin z}{(z+1)^7} dz$$

where C is a circle of radius 5 with centre 0 and the contour is oriented in an anticlockwise direction. [6]

(c) State the Residue theorem and use it to evaluate the contour integral of

$$g(z) = \frac{e^{iz}}{z^4 + z^2 + 1}$$

along the closed contour, oriented anticlockwise, consisting of $L_R = [-R, R]$ and C_R. Here L_R is the line between $-R$ and R and $C_R = \{|z| = R, \mathrm{Im}(z) \geqslant 0\}$ is a half-circle of radius R and centre 0, located above the real line.

Prove that

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{iz}}{z^4 + z^2 + 1} \, dz = 0.$$

Therefore, evaluate

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^4 + x^2 + 1} \, dx.$$

[10]

5A

The Fourier transform $\widetilde{f}(\omega)$ of a function $f(t)$ is defined by

$$\widetilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\, dt.$$

(a) Show that the Fourier transform of $f'(t)$ is given by $i\omega\widetilde{f}(\omega)$. Clearly state the assumptions you made about $f(t)$. [2]

(b) Consider the equation for forced damped harmonic motion

$$\frac{d^2y(t)}{dt^2} + 2\kappa\frac{dy(t)}{dt} + \Omega^2 y(t) = f(t),$$

where $\kappa, \Omega > 0$ are given constants and $f(t)$ is a given function.

Show that $\widetilde{y}(\omega)$ can be expressed as $\widetilde{y}(\omega) = \widetilde{h}(\omega)\widetilde{f}(\omega)$, and write down $\widetilde{h}(\omega)$. [2]

(c) Show that your expression in (b) can be inverted to find $y(t)$ as

$$y(t) = \int_{-\infty}^{\infty} G(t-\xi)f(\xi)d\xi,$$

where

$$G(t) = \int_{-\infty}^{\infty} \frac{s(\omega,t)}{(\omega-\omega_-)(\omega-\omega_+)}d\omega,$$

for some ω_-, ω_+ and $s(\omega,t)$ that you should determine. The convolution theorem can be used without proof. [3]

(d) Evaluate $G(t)$ for $t > 0$ by closing the contour and using the residue theorem for:

(i) $\Omega > \kappa$;

(ii) $\kappa > \Omega$;

(iii) $\kappa = \Omega$. [7]

What is the value of $G(t)$ for $t < 0$? Describe the behaviour of $G(t)$ as $t \to \infty$. [2]

(e) Use your results from parts (c) and (d) to determine $y(t)$ when $f(t) = \cos\kappa t$ and $\Omega = \kappa$. [4]

6C

(a) Show that any second-order tensor **T** can be written in the form

$$T_{ij} = S_{ij} + \epsilon_{ijk} u_k ,$$

where **S** is a symmetric second-order tensor and **u** is a vector. Find explicit expressions for S_{ij} and u_k in terms of T_{ij}. [5]

(b) Maxwell's equations for the electric and magnetic fields $\mathbf{E}(\boldsymbol{x}, t)$ and $\mathbf{B}(\boldsymbol{x}, t)$ in a vacuum can be written as

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 0, \qquad \boldsymbol{\nabla} \cdot \mathbf{B} = 0,$$

$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \qquad \boldsymbol{\nabla} \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = 0,$$

where c is a constant. Consider the second-order tensors $T^E_{ij} = \partial E_j / \partial x_i$ and $T^B_{ij} = \partial B_j / \partial x_i$. As in part (a), these can be written in terms of symmetric second-order tensors $\mathbf{S^E}$ and $\mathbf{S^B}$ and vectors $\mathbf{u^E}$ and $\mathbf{u^B}$, respectively.

(i) Calculate expressions for S^E_{ij}, S^B_{ij}, $\mathbf{u^E}$ and $\mathbf{u^B}$ in terms of **E** and **B**. [2]

(ii) Show that

$$\frac{\partial \mathbf{u^E}}{\partial t} = -\frac{c^2}{2} \nabla^2 \mathbf{B}.$$

[4]

(iii) Let V denote a constant closed volume with surface A. By applying the divergence theorem to a suitable integral expression, show that

$$\frac{\partial}{\partial t} \int_V \left(u^E_i + u^B_i \right) dV = \oint_A \left(S^E_{ij} - c^2 S^B_{ij} \right) dA_j \,.$$

[4]

(iv) Show further that

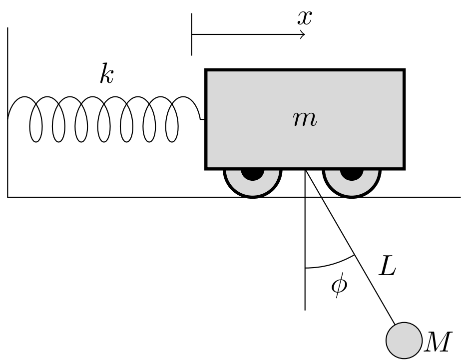
$$\frac{\partial}{\partial t} \int_V \lambda \, \mathrm{d}V = \oint_A (\mathbf{B} \times \mathbf{E}) \cdot \mathbf{dA} \,,$$

for some scalar quantity λ that should be determined in terms of **E**, **B** and c. [5]

7A

(a) Write down a general Lagrangian of a system with n degrees of freedom undergoing small oscillations, and state the polynomial equation for the normal frequencies. [5]

(b) A simple pendulum of mass M and length L is suspended from a cart of mass m that can oscillate on the end of a spring of force constant k, as shown in the figure. The cart is constrained to move in the horizontal direction only, and has a displacement $x(t)$ from its equilibrium position. The pendulum oscillates in the plane making angle $\phi(t)$ with the vertical direction.



(i) Assuming that the angle ϕ and displacement x remain small, write down the system's Lagrangian and the equations of motion for x and ϕ. [8]

(ii) Assuming that $m = M = L = g = 1$ and $k = 2$ (all in appropriate units), where g is the constant acceleration due to gravity, find the normal frequencies. For each normal frequency, find and describe the motion of the corresponding normal mode. [7]

8B

(a) Let G, G' be two finite groups and let $f : G \to G'$ be a group homomorphism. Let $a \in G$. Show that the order of $f(a)$ is at most the order of a. Show that if f is an isomorphism then a and $f(a)$ have the same order. [6]

(b) If $a, b \in G$ show that ab and ba have the same order. [7]

(c) Let G be a finite group where the order of each element is at most 2. Show that G is abelian. [7]

9B

(a) Let G be a group, and H_1 and H_2 two subgroups of G. Show that the claims (I) and (II) below are equivalent. [10]

(I) $H_1 \cap H_2 = \{1\}$ and any element $g \in G$ can be written as $g = h_1 h_2$, where $h_1 \in H_1$ and $h_2 \in H_2$.

(II) Any element $g \in G$ can be written in a *unique* way as $g = h_1 h_2$ where $h_1 \in H_1$ and $h_2 \in H_2$.

(b) Let H_1 be the group of matrices generated by $\left\{ \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] \right\}$ and let H_2 be the (cyclic) group generated by the single matrix $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Also let G be the smallest group containing H_1 and H_2. How many elements does G have? [5]

Show that G, H_1 and H_2 satisfy the condition (I). [5]

10A

(a) Let D be a representation of G; i.e. a homomorphism $D : G \to GL(n, \mathbb{C})$, where $GL(n, \mathbb{C})$ is the group of $n \times n$ invertible complex matrices. What does it mean for a vector subspace $W \subset \mathbb{C}^n$ to be an *invariant subspace* with respect to D? What does it mean for D to be *irreducible*? [4]

(b) Let $D_1 : G \to GL(n, \mathbb{C})$ be a representation, and define

$$D_2(g) = [D_1(g^{-1})]^\dagger$$

where † denotes the hermitian conjugate. Show that D_2 is a representation. [6]

(c) Suppose that W is an invariant subspace of $\mathbb{C}^n$ with respect to D_2. Show that $W_\perp$ is an invariant subspace of $\mathbb{C}^n$ with respect to D_1, where $W_\perp$ is the vector space of vectors orthogonal to W. Hence show that if D_1 is irreducible then D_2 must also be irreducible. [10]

END OF PAPER