UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Wednesday, 13 June, 2018 9:00 am to 12:00 pm

MATHEMATICS (2)

Before you begin read these instructions carefully:

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

You may submit answers to ***all*** *of section A, and to no more than* ***five*** *questions from section B.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on one side of the paper only and begin each answer on a separate sheet. *(For this purpose, your section A attempts should be considered as one single answer.)*

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Tie up ***all of your section A answers*** *in a single bundle, with a completed blue cover sheet.*

Each section B question has a number and a letter (for example, ***11S****). Answers to these questions must be tied up in* ***separate*** *bundles, marked* ***R, S, T, V, W, X, Y*** *or* ***Z*** *according to the letter affixed to each question.* ***Do not join the bundles together****. For each bundle, a blue cover sheet* ***must*** *be completed and attached to the bundle, with the correct letter* ***R, S, T, V, W, X, Y*** *or* ***Z*** *written in the section box.*

A ***separate*** *green master cover sheet listing all the questions attempted* ***must*** *also be completed. (Your section A answer may be recorded just as A: there is no need to list each individual short question.)*

Every cover sheet must bear your examination number and desk number.

Calculators are not permitted in this examination.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Green master cover sheet
Script paper

SPECIAL REQUIREMENTS
None.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1

Two intersecting lines are given by equations

$$\boldsymbol{r} = \begin{pmatrix}1\\0\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\-2\\1\end{pmatrix}, \qquad \boldsymbol{r} = \begin{pmatrix}1\\-8\\5\end{pmatrix} + \mu\begin{pmatrix}0\\-2\\1\end{pmatrix},$$

where λ and μ are real parameters. The lines lie in a common plane.

(a) What is the unit normal to this plane? [1]

(b) What is the shortest distance between the plane and the origin? [1]

2

Find all the solutions to

$$\sinh z = 0\,, \qquad [1]$$

and, separately, to

$$\cosh z + 1 = 0\,, \qquad [1]$$

where z is complex in both equations.

3

Consider the matrix

$$\begin{pmatrix} e^{\phi} & e^{-\phi} \\ e^{-\phi} & e^{\phi} \end{pmatrix},$$

where ϕ is a real number. Calculate the two eigenvalues of the matrix. [2]

4

Find the first two non-zero terms in the Taylor series expansion around $x = 0$ of the function

$$\frac{\ln(1+x^2)}{1+x^2}\,. \qquad [2]$$

5

Show that $f(xe^{-t})$ is a solution to the partial differential equation

$$\frac{\partial f}{\partial t} + x\frac{\partial f}{\partial x} = 0\,,$$

where f is any differentiable function of one variable. [1]

If $f = \ln(x)$ at $t = 0$, what is f for $t > 0$? [1]

6

Consider $f(r)$, a differentiable function of radius, $r = \sqrt{x^2 + y^2 + z^2}$. Find, in terms of f and df/dr,

(a) the Cartesian components of $\nabla \sin[f(r)]$, [1]

(b) the value of $\nabla \cdot [\hat{\boldsymbol{k}} f(r)]$, where $\hat{\boldsymbol{k}}$ is the unit vector along the z-axis. [1]

7

Solve the following differential equations for $y(x)$:

(a) $$\frac{dy}{dx} + y = e^{-x}\,,$$ [1]

with initial condition $y(0) = 0$, and

(b) $$\frac{dy}{dx} + (xy)^2 = 0\,,$$ [1]

with initial condition $y(0) = 1$.

8

For $\boldsymbol{F} = (\sin\theta)\boldsymbol{r}/r$, in spherical polar coordinates, calculate:

(a) the line integral

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r}\,,$$

where C is the unit circle lying in the x-y plane, centred on the origin, and traversed counterclockwise, [1]

(b) the surface integral

$$\int_S \boldsymbol{F} \cdot d\boldsymbol{S}\,,$$

where S is the surface of the unit sphere centred on the origin. [1]

9

Find the coordinates of the two stationary points of $u(x,\, y) = x^3 e^{-x^2-y^2}$ not located on the y-axis. [2]

10

A continuous random variable X takes values between 0 and π. Its normalised probability distribution is

$$f(x) = \alpha \sin x\,,$$

where α is a constant.

(a) What value does α take? [1]

(b) What is the mean of the distribution? [1]

SECTION B

11S

(a) Four arbitrary vectors are denoted by $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$. By considering the quadruple product $(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{c} \times \boldsymbol{d})$, or otherwise, prove that

$$\boldsymbol{d} = \frac{[\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}]}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]}\, \boldsymbol{a} - \frac{[\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{a}]}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]}\, \boldsymbol{b} + \frac{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}]}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]}\, \boldsymbol{c}\,,$$

where $[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}] = (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} \neq 0$. [6]

(b) A point in space has position vector $\boldsymbol{r}_0$ and a line is given by the equation $\boldsymbol{r} - \boldsymbol{a} = \lambda \widehat{\boldsymbol{t}}$, where λ is a real parameter and $\widehat{\boldsymbol{t}}$ is a unit vector. Show that the perpendicular distance from the point to the line is $|\widehat{\boldsymbol{t}} \times (\boldsymbol{r}_0 - \boldsymbol{a})|$. [3]

Hence find the closest distance between the line given by

$$\boldsymbol{r} = \frac{\lambda}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

and the parabola given by

$$\boldsymbol{r} = \begin{pmatrix} \mu \\ (1 + 2\sqrt{3})\mu \\ \mu^2 - 4 \end{pmatrix},$$

where μ is a real parameter. [5]

[*Hint: Consider* $d(\ell^2)/d(\mu^2)$, *where* ℓ *is the perpendicular distance between any point on the parabola and the line.*]

(c) Consider the vector equation

$$2\boldsymbol{x} + \widehat{\boldsymbol{n}} \times \boldsymbol{x} + \widehat{\boldsymbol{n}}\,(\widehat{\boldsymbol{n}} \cdot \boldsymbol{x})^2 = \boldsymbol{b}\,,$$

in which $\boldsymbol{x}$ is an unknown vector, $|\widehat{\boldsymbol{n}}| = 1$, and $\widehat{\boldsymbol{n}} \cdot \boldsymbol{b} = -1$. Find $\boldsymbol{x}$. [6]

[*Hint: Take the scalar and vector products of the equation with* $\widehat{\boldsymbol{n}}$. *Do* not *write out the equation in components!*]

12X

A scalar field in two dimensions can be represented either in terms of Cartesian coordinates as $f(x, y)$ or in terms of polar coordinates as $g(r, \phi)$. Thus at every point

$$f(x, y) = g(r, \phi)\,.$$

The relationship between the coordinate systems is as usual:

$$x = r\cos\phi\,, \qquad y = r\sin\phi\,.$$

You are encouraged to use the shorthand notation:

$$\left(\frac{\partial f}{\partial x}\right)_y = f_x\,, \qquad \left(\frac{\partial f}{\partial \phi}\right)_r = f_\phi\,, \quad \text{etc.}$$

and to write $\cos\phi = c\,, \quad \sin\phi = s$.

(a) Show that $(\partial r/\partial x)_y = \cos\phi$ and find a similar formula for $(\partial r/\partial y)_x$. [2]

(b) Show that $(\partial \phi/\partial x)_y = -\sin\phi/r$ and find a similar formula for $(\partial \phi/\partial y)_x$. [2]

(c) Hence show that

$$f_x = g_r \cos\phi - g_\phi (\sin\phi)/r$$

and find a similar formula (involving only r, ϕ and the partial derivatives of g) for f_y. [4]

(d) Show that

$$f_{xx} = g_{rr}c^2 - g_{r\phi}\frac{2sc}{r} + g_\phi \frac{2sc}{r^2} + g_r \frac{s^2}{r} + g_{\phi\phi}\frac{s^2}{r^2},$$

and find a similar expression for f_{yy}. Hence determine a formula in polar coordinates for $\nabla^2 f = f_{xx} + f_{yy}$. [5]

(e) Take the particular case $g = r^2 \sin 2\phi$.

(i) Calculate f and $\boldsymbol{\nabla} f$ in terms of x and y. [2]

(ii) Sketch $\boldsymbol{\nabla} f$ as a vector field in the positive quadrant of the x-y plane. Include the contours of constant f. [3]

(iii) For the point $(x, y) = (1, 2)$ calculate the gradient of f in the direction parallel to $(1, 1)$ (i.e. the directional derivative). [2]

13R

(a) Consider the vector fields $\boldsymbol{F} = (-y,\, x)$ and $\boldsymbol{G} = (2xy^2,\, 2yx^2)$. Evaluate their line integrals along the following closed paths in the x-y plane:

(i) The four sides of the unit square with corners at $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$, starting at $(1,0)$ and proceeding counterclockwise; [4]

(ii) The circumference of the unit circle centred on the origin, starting at $(1,0)$ and proceeding counterclockwise. [4]

Are $\boldsymbol{F}$ and $\boldsymbol{G}$ conservative? If so, write the field(s) as $\nabla\Phi$ where Φ is a scalar potential, which you should find. [4]

(b) A directed three-dimensional curve, Γ, is given parametrically by

$$x = \cos 6t, \qquad y = \sin 6t, \qquad z = 8t,$$

with the parameter t increasing from 0 to $\pi/12$.

(i) Find the Cartesian coordinates of the start and end points of Γ and sketch the curve in three-dimensional space. [3]

(ii) Evaluate the line integral $\int_\Gamma \Phi(x, y, z) ds$ where ds is an infinitesimal arclength and $\Phi(x, y, z) = xy$. [5]

14V

A factory produces good (G) and defective (D) balls with probability p and $1-p$, respectively. To test if a ball is good or defective, it is rolled along the x-axis starting from the origin. The ball then comes to rest at some point from the following discrete set of x-coordinates: $x_k = x_0 + k\delta$, where x_0 and δ are positive parameters and $k = 0, 1, \ldots, n$, where n is an integer. A ball stops at x_k with probability g_k if it is a good ball and with probability d_k if it is defective. In an experiment, a ball is produced and tested.

(a) Consider the particular case with $n = 2$, so that the discrete random variable X, the coordinate of the stopping point, takes values $X \in \{x_0, x_1, x_2\}$.

(i) Using notation, such as $G \cap x_k$ (i.e. a ball is good and stops at $X = x_k$), find the sample space for the experiment assuming $g_k > 0$, $d_k > 0$ and $0 < p < 1$. [2]

(ii) What is the probability that a good ball is produced and it stops at x_0? [1]

(iii) What is the probability that a ball stops at x_0? [2]

(iv) Find the probability $P(D|x_1)$. [2]

(v) Given that a ball stops at $X > x_0$, find the probability that it is a good ball. [3]

(b) Consider the general case with $n \geqslant 2$.

(i) Find the probability that a ball stops at $X > x_k$. [3]

(ii) Given that a ball stops at $X \leqslant x_k$, find the probability that it is a good ball. [3]

(iii) If all the values of d_k are equal to each other and $n = 98$, find the minimal value of p for which $P(G|x_0) = 0.99$. [4]

15Y

(a) Find the general solutions to the following equations

(i) $$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 2x^2\,,$$ [7]

(ii) $$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{-x}\,.$$ [7]

(b) Consider the equation

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} + \frac{1}{x}\,y = 0\,, \qquad (\dagger)$$

where $x > 0$. Show that if $y(x) = xu(x)$, then $u(x)$ is the solution of

$$x\frac{d^2u}{dx^2} + \frac{du}{dx} = 0\,.$$

Hence find the general solution of equation (†). [6]

16W

Let fixed points A and B in three-dimensional space be given by non-zero position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively, and let $\boldsymbol{r} = (x, y, z)$.

(a) Simplify

(i) $\nabla \cdot (2(\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{r} + \boldsymbol{a})$, [2]

(ii) $\nabla \times ((\boldsymbol{a} \times \boldsymbol{b}) + \boldsymbol{r})$, [2]

(iii) $\nabla \cdot (\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{r}))$, [3]

(iv) $\nabla \times (\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{r}))$. [3]

(b) Calculate the flux of the vector field $\boldsymbol{F} = \boldsymbol{a} \times \boldsymbol{b} + (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{r}$ through

(i) the triangle $OABO$ where O denotes the origin; [5]

(ii) the closed hemisphere with curved surface and base given by

$$x^2 + y^2 + z^2 = R^2 \,, \quad 0 \leqslant z \leqslant R \,,$$

and

$$x^2 + y^2 \leqslant R^2 \,, \quad z = 0 \,,$$

respectively, where the parameter $R > 0$. [5]

17R

Consider the set of simultaneous equations

$$\begin{cases} x + \mu y &= b_1\,, \\ x - y + 3z &= b_2\,, \\ 2x - 2y + \mu z &= b_3\,, \end{cases}$$

where μ, b_1, b_2 and b_3 are real parameters.

(a) Write down the above set in matrix form

$$\boldsymbol{A}\boldsymbol{r} = \boldsymbol{b} \qquad (\dagger)$$

specifying all the elements of matrix $\boldsymbol{A}$ and column vectors $\boldsymbol{r}$ and $\boldsymbol{b}$. [2]

(b) Demonstrate that equation (†) can be written as

$$x\boldsymbol{c}_1 + y\boldsymbol{c}_2 + z\boldsymbol{c}_3 = \boldsymbol{b}\ ,$$

where $\boldsymbol{c}_1$, $\boldsymbol{c}_2$ and $\boldsymbol{c}_3$ are column vectors that should be specified. [2]

(c) Consider the particular case $\boldsymbol{b} = \boldsymbol{0}$.

(i) Prove that a non-trivial solution to (†) exists only if $[\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3] = 0$. Recall that $[\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3]$ is the scalar triple product $(\boldsymbol{c}_1 \times \boldsymbol{c}_2) \cdot \boldsymbol{c}_3$. [2]

(ii) Find all values of μ for which equation (†) has a non-trivial solution. [2]

(iii) Hence, find all non-trivial solutions of equation (†) and interpret them geometrically. [6]

(d) Consider another particular case $\boldsymbol{b} = (1,\ \lambda,\ 0)^T$ with real parameter λ and with $\mu = -1$. Prove that a non-trivial solution exists only if

$$[\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{b}] = [\boldsymbol{c}_2, \boldsymbol{c}_3, \boldsymbol{b}] = [\boldsymbol{c}_1, \boldsymbol{c}_3, \boldsymbol{b}] = 0\,.$$

Hence, find the value of λ, solve equation (†) and interpret the solution geometrically. [6]

18T

(a) Write down the Fourier series expansion of an arbitrary 2π-periodic function f together with expressions for the coefficients, and state Parseval's identity. [5]

(b) For $0 \leqslant r < 1$ and $-\pi \leqslant \theta < \pi$, let

$$K(r,\theta) = \frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos n\theta,$$

and for an arbitrary 2π-periodic function f, let

$$y_f(r,\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} K(r,\theta - t) f(t)\, dt\,.$$

Prove that

$$y_f(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)\,, \qquad (\dagger)$$

and find a_n and b_n in terms of f. [5]

(c) Find the coefficients a_n and b_n when

$$f(t) = f_0(t) = \begin{cases} -1, & -\pi \leqslant t < 0, \\ 1, & 0 \leqslant t < \pi; \end{cases}$$

and write down an expression for $y_{f_0}(r,\theta)$ in the form ($\dagger$). [5]

(d) Let D and dA be the unit disc and elementary area of integration, respectively. By considering r, θ as polar coordinates, and using Parseval's identity, prove that

$$\int_D [y_{f_0}]^2\, dA = \frac{8}{\pi} \sum_{n \text{ odd}}^{\infty} \frac{1}{n^p(n+1)}\,,$$

where p is a real positive number which you should find. [5]

19W*

(a) Let $f(x, y, z) = axy + z$ be subject to the constraint $x^2 + y^2 + z - x = 0$, where a is a parameter.

(i) Using the method of Lagrange multipliers find the stationary points of $f(x, y, z)$. [5]

(ii) By considering f as function of two independent variables, i.e. $f(x, y, z) = f(x, y, z(x, y))$, use the properties of the Hessian to determine the types of the stationary points found in (i). [5]

(b) The function $f(n_0, n_1, n_2, \ldots) = -\sum_{k=0}^{\infty}[n_k \ln(n_k) - n_k]$ of an infinite number of positive variables is subject to two constraints,

$$\sum_{k=0}^{\infty} n_k = N \quad \text{and} \quad \sum_{k=0}^{\infty} E_0 \left(\frac{1}{2} + k\right) n_k = E \ ,$$

where N, E_0 and E are positive constants.

Using the method of Lagrange multipliers show that the stationary point of $f(n_0, n_1, n_2, \ldots)$ subject to the above constraints occurs when

$$n_k = 2N \sinh\left(\frac{\beta E_0}{2}\right) e^{-\beta E_0 \left(\frac{1}{2}+k\right)} \quad \text{for all } k \ , \qquad [4]$$

where β is a Lagrange multiplier. Show further that

$$E = \frac{NE_0}{2} \coth\left(\frac{\beta E_0}{2}\right) \ . \qquad [6]$$

20Y*

(a) Consider the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 1. \qquad (\dagger)$$

(i) By making the change of variables

$$\xi = x + y, \qquad \eta = x - y,$$

show that the equation reduces to

$$4\frac{\partial^2 u}{\partial \xi \partial \eta} = 1.$$ [4]

Hence determine the most general form for u, the solution to (†). [4]

(ii) Suppose that the solution to (†) obeys the boundary conditions $u = \partial u/\partial y = 0$ on $y = 0$ and takes the form

$$u = ax^2 + bxy + cy^2 + d\,,$$

where a, b, c, d are real coefficients. Determine the coefficients. [4]

(b) The equation $3y = z^3 + 3xz$ defines z implicitly as a function of x and y. Evaluate the second partial derivatives of z with respect to x and y to verify that z is a solution of

$$\frac{\partial^2 z}{\partial x^2} + x\frac{\partial^2 z}{\partial y^2} = 0.$$ [8]

END OF PAPER