Before you begin read these instructions carefully:

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

You may submit answers to all of section A, and to no more than five questions from section B.

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on one side of the paper only and begin each answer on a separate sheet. (For this purpose, your section A attempts should be considered as one single answer.)

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Tie up all of your section A answers in a single bundle, with a completed blue cover sheet.

Each section B question has a number and a letter (for example, 11Z). Answers to these questions must be tied up in separate bundles, marked R, S, T, V, W, X, Y or Z according to the letter affixed to each question. Do not join the bundles together. For each bundle, a blue cover sheet must be completed and attached to the bundle, with the correct letter R, S, T, V, W, X, Y or Z written in the section box.

A separate green master cover sheet listing all the questions attempted must also be completed. (Your section A answer may be recorded just as A: there is no need to list each individual short question.)

Every cover sheet must bear your examination number and desk number.

Calculators are not permitted in this examination.

STATIONERY REQUIREMENTS

6 blue cover sheets and treasury tags
Green master cover sheet
Script paper

SPECIAL REQUIREMENTS

None.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.
SECTION A

1

(a) Solve the equation $2^x = 10^y$, for $x$. \[1\]
(b) Calculate $x$ to two significant figures for the case $y = 23.8$, given $\log_{10} 2 \approx 0.301$. \[1\]

2

Solve for $x$:
(a) $2x^3 + 9x^2 - 6x = 5$, \[1\]
(b) $x^2 + 3x + 1 < 0$. \[1\]

3

Calculate the values of $t$ and $\phi$ for the points of intersection of the two curves

\[ x = 2 \cos \phi, \quad y = \sin \phi; \quad 0 \leq \phi < 2\pi \]
and
\[ x = -2t, \quad y = 1 + t; \quad -\infty < t < \infty. \] \[2\]

4

(a) Calculate $\int_{0}^{\pi/6} x \sin 3x \, dx$. \[1\]
(b) Integrate $(\sin 2x)e^{\sin^2 x}$ with respect to $x$. \[1\]
5

\( O \) is the centre of a circle of radius \( r \) as shown in the diagram.

(a) Given that the area \( A \) is twice the area \( B \), calculate \( \theta \). \([1]\]
(b) In terms of the area \( B \), calculate the area of the triangle \( OXY \). \([1]\)

\[
\begin{align*}
A & \quad O \\
\quad r \\
\theta & \quad X \\
\quad B \\
\quad Y
\end{align*}
\]

6

(a) Find the length of the diagonal of the unit cube. \([1]\)
(b) Using a scalar product or otherwise, find the acute angle between the diagonals of the unit cube. \([1]\)

7

(a) Calculate to three decimal places \((1.01)^9\). \([1]\)
(b) Calculate \( \sum_{n=-1}^{\infty} \left( \frac{1}{3} \right)^n \). \([1]\)

8

Calculate \( x \) and \( y \) for the stationary point of the function \( y = e^x \cos x \) in the range \( 0 < x < \pi/2 \). \([2]\)
Consider the circle defined by the equation:

\[ x^2 + y^2 + x + 2y = 1. \]

(a) Find the coordinates of the centre. [1]
(b) Find its radius. [1]

(a) Find the coordinates of the point of intersection of the two curves given by the functions:

\[ y = x^3 + 3x \quad ; \quad y = 3x^2 + 1. \] [1]

(b) At the point of intersection, the curves defined in part (a) have a common tangent line. Calculate the coordinates of intersection of this common tangent line with the \( y \)-axis. [1]
(a) Find the moduli and arguments of $z$ and $w$, where:

$$z = \frac{1}{2}(1 + i)e^{i\pi/6}, \quad w = \ln z.$$  

(b) Find all the solutions to the equation

$$\tanh z = -i.$$  

(c) Use complex numbers to show that

$$\cos 5\theta = 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta.$$
(a) Consider the double integral

\[ I = \int \int_D xe^{-xy} \, dx \, dy, \]

where the area of integration \( D \) is defined by the following inequalities, \( x \geq 0 \) and \( x/2 \leq y \leq x \).

(i) Sketch the area of integration in the \( x-y \) plane. \([1]\)

(ii) Evaluate \( I \) by integrating first over \( y \). \([5]\)

(iii) By changing the order of integration in \( I \) or otherwise, evaluate the following integral,

\[ \int_0^\infty \left( e^{-x^2} - e^{-2x^2} \right) \frac{dx}{x^2}. \] \([6]\)

(b) The surface of an ellipsoid is defined by the following equation,

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \]

where \( a, b \) and \( c \) are positive parameters.

(i) Express the volume of the ellipsoid as a triple integral and calculate this integral. \([4]\)

(ii) Evaluate

\[ \iiint_V \left( x^4 y^2 z + xy^2 z^4 \right) e^{-(x/a)^2-(y/b)^2-(z/c)^2} \, dx \, dy \, dz, \]

where \( V \) is the interior region of the ellipsoid. \([4]\).
13Y

Solve the following differential equations, giving \( y \) explicitly for (a) and (b).

(a) \( x^2 \frac{dy}{dx} = 1 - xy \). \[3\]

(b) \( \frac{dy}{dx} = e^{3x} - y \). \[3\]

(c) \( \frac{dy}{dx} = \frac{5x + 4y}{8y^3 - 4x} \). \[6\]

(d) \( (x + \sqrt{xy}) \frac{dy}{dx} = y \), for \( x > 0 \). \[8\]

14X

A function of two variables is defined by

\[ f(x, y) = (1 - 4xy)\exp[-(x^2 + y^2)]. \]

(a) Give the equation of the contour on which \( f = 0 \). \[2\]

(b) Sketch this contour in the \( x \)-\( y \) plane and mark regions for which \( f > 0 \) and \( f < 0 \). \[2\]

(c) Find the coordinates of the stationary points (values of \( f \) are not required) and mark the points on your sketch. \[6\]

(d) Determine the character of each of the stationary points, justifying your answers. \[6\]

(e) Complete the contour map of \( f(x, y) \). \[4\]
(a) State Taylor’s theorem for the expansion about $x = a$ of a function that is $n$ times differentiable keeping the first $n$ terms, together with an expression for the remainder term $R_n$. [4]

(b) Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$, where $a_n$ are real coefficients. Given that $f(0) \neq 0$, find in terms of the $a_n$ the first three terms in the Taylor series expansion about $x = 0$ of the function $1/f(x)$. [6]

(c) Find the Taylor series expansion about $x = 0$ up to and including the term proportional to $x^4$ of the following function:

$$\frac{\cosh x}{\sqrt{1 + x^2}}.$$ You may use standard Taylor series expansions without proof. [6]

(d) Find the first three non-zero terms in a series approximation of $\ln(1 + x + 2x^2) - \ln x^2$ valid for $x \to \infty$. [4]
A cannon fixed at the origin in the $x$-$y$ plane fires discrete particles. The $i^{th}$ particle moves within the plane along a straight line which makes a random angle $\Theta = \theta_i$ with the $x$-axis and collides with a screen at $(x_0, y_i)$, where the parameter $x_0 > 0$ defines the position of the screen on the $x$-axis. The values of $\theta_i$ are uniformly distributed in the range, $-\pi/2 \leq \theta_i \leq \pi/2$.

(a) Find $y_i$ in terms of $\theta_i$ and $x_0$. [2]

(b) Find the probability density function, $g(\theta)$, and cumulative distribution function, $G(\theta) = P(\Theta \leq \theta)$, where $P(Z)$ stands for the probability of event $Z$. [2]

(c) Let $Y$ be a continuous random variable denoting the $y$-coordinate of the collision points. By using results found in (a) and (b), find the cumulative distribution function, $F(y) = P(Y \leq y)$. [2]

(d) Hence show that the probability density function, $f(y)$, is given by

$$f(y) = \frac{x_0}{\pi (x_0^2 + y^2)}.$$ [2]

(e) Show that the standard deviation of the $y$-coordinate of the collision position has no mathematically defined value. [3]

(f) The cannon is adjusted such that its firing angle, $\theta$, becomes uniformly distributed between $-\pi/6$ and $+\pi/3$. Find the expectation value of $Y$, $E[Y]$. [5]

(g) The position of the screen is adjusted so that its distance from the cannon instantaneously halves each time a particle is fired (still firing between $-\pi/6$ and $+\pi/3$); that is, the first particle hits the screen when its $x$ coordinate is $x_0/2$. Find the expectation value of the sum of the collision $y$-coordinates, $E[\sum_{i=1}^\infty Y_i]$, of an infinite sequence of fired particles. [4]
17Z

(a) Evaluate the indefinite integral
\[ \int \frac{dx}{\cos^2 x (\tan^3 x - \tan x)}. \] [6]

(b) Consider
\[ I_n = \int_0^\infty x^{n-1} e^{-x} \, dx, \]
where \( n \geq 1 \) is an integer. By deriving and solving a recursion relation for \( I_n \), or otherwise, show that \( I_n = (n-1)! \) [6]

(c) Suppose that
\[ E_n = \int_0^\infty \frac{dx}{(1 + x^2)^n}, \]
where \( n \geq 1 \) is an integer.
Find a recursion relation expressing \( E_{n+1} \) in terms of \( E_n \). Hence evaluate
\[ \int_0^\infty \frac{dx}{(1 + x^2)^4}. \] [8]

18R

(a) For the matrix
\[ A = \begin{pmatrix} 1 & -4 & 7 \\ -4 & 4 & -4 \\ 7 & -4 & 1 \end{pmatrix}, \]
calculate \( \text{Det}(A) \) and \( \text{Tr}(A) \). Further, calculate \( \text{Det}(A^T) \) and \( \text{Det}(A^2) \). [4]

(b) Calculate the eigenvalues \( \lambda_i \) and the corresponding normalized eigenvectors \( v_i \) of \( A \).
Verify that the eigenvectors are mutually orthogonal. [10]

(c) By expressing an arbitrary vector \( r \) in terms of the eigenvectors of \( A \), or otherwise, find a non-zero vector \( u \) such that, for any \( r \), we have \( (Ar) \cdot u = 0 \). [6]
(a) (i) A real function \( f(x) \) is defined on an interval containing the interior point \( x_0 \).
   Explain what is meant by the statement that the function \( f(x) \) is
   
   (1) continuous at \( x = x_0 \), \[1\]
   (2) differentiable at \( x = x_0 \). \[1\]

(ii) For \( m = 0, 1, 2 \), the function \( f_m(x) \) is defined as
   \[
   f_m(x) = \begin{cases} 
   x^m \sin \frac{1}{x} & \text{if } x \neq 0; \\
   0 & \text{if } x = 0.
   \end{cases}
   \]

For each \( m = 0, 1, 2 \), determine whether the function \( f_m(x) \) is
   (1) continuous at \( x = 0 \),
   (2) differentiable at \( x = 0 \).
   
   Justify your answer. \[10\]

(b) Using the comparison or the ratio test, or otherwise, determine whether the following series converge or diverge. You may use without proof standard results relating to the series \( \sum_{n=1}^{\infty} n^{-p} \), where \( p > 0 \).

(i) \[
\sum_{n=1}^{\infty} \frac{n!}{2^n} \] \[4\]

(ii) \[
\sum_{n=1}^{\infty} \left( \sqrt{n^4 + a^2} - n^2 \right), \quad a > 0. \] \[4\]
(a) Evaluate the integral 
\[ \int_0^1 \frac{1}{1 + \alpha^2 x^2} \, dx, \]
where \( \alpha \) is a real parameter. By differentiating with respect to \( \alpha \), show that 
\[ \int_0^1 \frac{x^2}{(1 + 3x^2)^2} \, dx = \frac{\pi \sqrt{3}}{54} - \frac{1}{24}. \] [8]

(b) Suppose that 
\[ E(\alpha) = \int_0^\infty \frac{\ln(1 + \alpha^2 x^2)}{1 + x^2} \, dx, \]
where \( \alpha \) is a real parameter. Demonstrate that 
\[ \frac{dE}{d\alpha} = \frac{\pi}{1 + \alpha}. \] [8]

Solve this equation for \( E(\alpha) \) and hence show that 
\[ \int_0^\infty \frac{\ln(1 + x^2)}{1 + x^2} \, dx = \pi \ln 2. \] [4]