UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Monday, 11 June, 2018 9:00 am to 12:00 pm

MATHEMATICS (1)

Before you begin read these instructions carefully:

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

*You may submit answers to **all** of section A, and to no more than **five** questions from section B.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

***Write on one side of the paper only and begin each answer on a separate sheet.** (For this purpose, your section A attempts should be considered as one single answer.)*

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

*Tie up **all of your section A answers** in a single bundle, with a completed blue cover sheet.*

*Each section B question has a number and a letter (for example, **11Z**). Answers to these questions must be tied up in **separate** bundles, marked **R, S, T, V, W, X, Y** or **Z** according to the letter affixed to each question. **Do not join the bundles together**. For each bundle, a blue cover sheet **must** be completed and attached to the bundle, with the correct letter **R, S, T, V, W, X, Y** or **Z** written in the section box.*

*A **separate** green master cover sheet listing all the questions attempted **must** also be completed. (Your section A answer may be recorded just as A: there is no need to list each individual short question.)*

Every cover sheet must bear your examination number and desk number.

Calculators are not permitted in this examination.

STATIONERY REQUIREMENTS	***SPECIAL REQUIREMENTS***
6 blue cover sheets and treasury tags	*None.*
Green master cover sheet	
Script paper	

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1

(a) Solve the equation $2^x = 10^y$, for x. [1]

(b) Calculate x to two significant figures for the case $y = 23.8$, given $\log_{10} 2 \simeq 0.301$. [1]

2

Solve for x:

(a) $2x^3 + 9x^2 - 6x = 5\,,$ [1]

(b) $x^2{+}3x{+}1 < 0\,.$ [1]

3

Calculate the values of t and ϕ for the points of intersection of the two curves

$$x = 2\cos\phi, \quad y = \sin\phi \quad ; \quad 0 \leqslant \phi < 2\pi$$

$$\text{and} \quad x = -2t, \quad y = 1 + t \quad ; \quad -\infty < t < \infty.$$

[2]

4

(a) Calculate $\displaystyle\int_0^{\pi/6} x \sin 3x \, \mathrm{d}x$. [1]

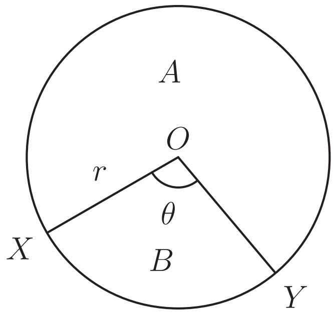
(b) Integrate $(\sin 2x)e^{\sin^2 x}$ with respect to x. [1]

5

O is the centre of a circle of radius r as shown in the diagram.

(a) Given that the area A is twice the area B, calculate θ. [1]

(b) In terms of the area B, calculate the area of the triangle OXY. [1]



6

(a) Find the length of the diagonal of the unit cube. [1]

(b) Using a scalar product or otherwise, find the acute angle between the diagonals of the unit cube. [1]

7

(a) Calculate to three decimal places $(1.01)^9$. [1]

(b) Calculate $\sum_{n=-1}^{\infty} (\frac{1}{3})^n$. [1]

8

Calculate x and y for the stationary point of the function $y = e^x \cos x$ in the range $0 < x < \pi/2$. [2]

9

Consider the circle defined by the equation:

$$x^2 + y^2 + x + 2y = 1\,.$$

(a) Find the coordinates of the centre. [1]

(b) Find its radius. [1]

10

(a) Find the coordinates of the point of intersection of the two curves given by the functions:

$$y = x^3 + 3x \quad ; \quad y = 3x^2 + 1\,.$$ [1]

(b) At the point of intersection, the curves defined in part (a) have a common tangent line. Calculate the coordinates of intersection of this common tangent line with the y-axis. [1]

SECTION B

11Z

(a) Find the moduli and arguments of z and w, where:

$$z = \frac{1}{2}(1+i)e^{i\pi/6}, \qquad w = \ln z\,.$$ [10]

(b) Find all the solutions to the equation

$$\tanh z = -i\,.$$ [5]

(c) Use complex numbers to show that

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta\,.$$ [5]

12W

(a) Consider the double integral

$$I = \iint_D x e^{-xy} dx\, dy \ ,$$

where the area of integration D is defined by the following inequalities, $x \geqslant 0$ and $x/2 \leqslant y \leqslant x$.

(i) Sketch the area of integration in the x-y plane. [1]

(ii) Evaluate I by integrating first over y. [5]

(iii) By changing the order of integration in I or otherwise, evaluate the following integral,

$$\int_0^\infty \left(e^{-x^2} - e^{-2x^2} \right) \frac{dx}{x^2} \ .$$ [6]

(b) The surface of an ellipsoid is defined by the following equation,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \ ,$$

where a, b and c are positive parameters.

(i) Express the volume of the ellipsoid as a triple integral and calculate this integral. [4]

(ii) Evaluate

$$\iiint_V \left(x^4 y^2 z + x y^2 z^4 \right) e^{-(x/a)^2-(y/b)^2-(z/c)^2} dx\, dy\, dz \ ,$$

where V is the interior region of the ellipsoid. [4]

13Y

Solve the following differential equations, giving y explicitly for (a) and (b).

(a) $$x^2\frac{dy}{dx} = 1 - xy\,.$$ [3]

(b) $$\frac{dy}{dx} = e^{3x} - y\,.$$ [3]

(c) $$\frac{dy}{dx} = \frac{5x+4y}{8y^3-4x}\,.$$ [6]

(d) $$(x+\sqrt{xy})\frac{dy}{dx} = y\,, \qquad \text{for } x>0.$$ [8]

14X

A function of two variables is defined by

$$f(x,y) = (1-4xy)\exp[-(x^2+y^2)].$$

(a) Give the equation of the contour on which $f=0$. [2]

(b) Sketch this contour in the x-y plane and mark regions for which $f>0$ and $f<0$. [2]

(c) Find the coordinates of the stationary points (values of f are not required) and mark the points on your sketch. [6]

(d) Determine the character of each of the stationary points, justifying your answers. [6]

(e) Complete the contour map of $f(x,y)$. [4]

15T

(a) State Taylor's theorem for the expansion about $x = a$ of a function that is n times differentiable keeping the first n terms, together with an expression for the remainder term R_n. [4]

(b) Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$, where a_n are real coefficients. Given that $f(0) \neq 0$, find in terms of the a_n the first three terms in the Taylor series expansion about $x = 0$ of the function $1/f(x)$. [6]

(c) Find the Taylor series expansion about $x = 0$ up to and including the term proportional to x^4 of the following function:

$$\frac{\cosh x}{\sqrt{1+x^2}} \,.$$

You may use standard Taylor series expansions without proof. [6]

(d) Find the first three non-zero terms in a series approximation of $\ln(1+x+2x^2) - \ln x^2$ valid for $x \to \infty$. [4]

16V

A cannon fixed at the origin in the x-y plane fires discrete particles. The i^{th} particle moves within the plane along a straight line which makes a random angle $\Theta = \theta_i$ with the x-axis and collides with a screen at (x_0, y_i), where the parameter $x_0 > 0$ defines the position of the screen on the x-axis. The values of θ_i are uniformly distributed in the range, $-\pi/2 \leqslant \theta_i \leqslant \pi/2$.

(a) Find y_i in terms of θ_i and x_0. [2]

(b) Find the probability density function, $g(\theta)$, and cumulative distribution function, $G(\theta) = P(\Theta \leqslant \theta)$, where $P(Z)$ stands for the probability of event Z. [2]

(c) Let Y be a continuous random variable denoting the y-coordinate of the collision points. By using results found in (a) and (b), find the cumulative distribution function, $F(y) = P(Y \leqslant y)$. [2]

(d) Hence show that the probability density function, $f(y)$, is given by [2]

$$f(y) = \frac{x_0}{\pi(x_0^2 + y^2)} \ .$$

(e) Show that the standard deviation of the y-coordinate of the collision position has no mathematically defined value. [3]

(f) The cannon is adjusted such that its firing angle, θ, becomes uniformly distributed between $-\pi/6$ and $+\pi/3$. Find the expectation value of Y, $\mathrm{E}[Y]$. [5]

(g) The position of the screen is adjusted so that its distance from the cannon instantaneously halves each time a particle is fired (still firing between $-\pi/6$ and $+\pi/3$); that is, the first particle hits the screen when its x coordinate is $x_0/2$. Find the expectation value of the sum of the collision y-coordinates, $\mathrm{E}\left[\sum_{i=1}^{\infty} Y_i\right]$, of an infinite sequence of fired particles. [4]

17Z

(a) Evaluate the indefinite integral

$$\int \frac{dx}{\cos^2 x(\tan^3 x - \tan x)} \,. \qquad [6]$$

(b) Consider

$$I_n = \int_0^\infty x^{n-1} \mathrm{e}^{-x} \, dx \,,$$

where $n \geqslant 1$ is an integer. By deriving and solving a recursion relation for I_n, or otherwise, show that $I_n = (n-1)!$ [6]

(c) Suppose that

$$E_n = \int_0^\infty \frac{dx}{(1+x^2)^n} \,,$$

where $n \geqslant 1$ is an integer.

Find a recursion relation expressing E_{n+1} in terms of E_n. Hence evaluate

$$\int_0^\infty \frac{dx}{(1+x^2)^4} \,. \qquad [8]$$

18R

(a) For the matrix

$$\boldsymbol{A} = \begin{pmatrix} 1 & -4 & 7 \\ -4 & 4 & -4 \\ 7 & -4 & 1 \end{pmatrix} ,$$

calculate $\mathrm{Det}(\boldsymbol{A})$ and $\mathrm{Tr}(\boldsymbol{A})$. Further, calculate $\mathrm{Det}(\boldsymbol{A}^T)$ and $\mathrm{Det}(\boldsymbol{A}^2)$. [4]

(b) Calculate the eigenvalues λ_i and the corresponding normalized eigenvectors $\boldsymbol{v}_i$ of $\boldsymbol{A}$. Verify that the eigenvectors are mutually orthogonal. [10]

(c) By expressing an arbitrary vector $\boldsymbol{r}$ in terms of the eigenvectors of $\boldsymbol{A}$, or otherwise, find a non-zero vector $\boldsymbol{u}$ such that, for any $\boldsymbol{r}$, we have $(\boldsymbol{A}\boldsymbol{r}) \cdot \boldsymbol{u} = 0$. [6]

19T*

(a) (i) A real function $f(x)$ is defined on an interval containing the interior point x_0. Explain what is meant by the statement that the function $f(x)$ is

(1) continuous at $x = x_0$, [1]

(2) differentiable at $x = x_0$. [1]

(ii) For $m = 0, 1, 2$, the function $f_m(x)$ is defined as

$$f_m(x) = \begin{cases} x^m \sin \frac{1}{x}, & x \neq 0\,; \\ 0, & x = 0\,. \end{cases}$$

For each $m = 0, 1, 2$, determine whether the function $f_m(x)$ is

(1) continuous at $x = 0$,

(2) differentiable at $x = 0$.

Justify your answer. [10]

(b) Using the comparison or the ratio test, or otherwise, determine whether the following series converge or diverge. You may use without proof standard results relating to the series $\sum_{n=1}^{\infty} n^{-p}$, where $p > 0$.

(i) $$\sum_{n=1}^{\infty} \frac{n!}{2^n}\,,$$ [4]

(ii) $$\sum_{n=1}^{\infty} \left(\sqrt{n^4 + a^2} - n^2 \right), \quad a > 0\,.$$ [4]

20S*

(a) Evaluate the integral

$$\int_0^1 \frac{1}{1+\alpha^2 x^2}\,dx,$$

where α is a real parameter. By differentiating with respect to α, show that

$$\int_0^1 \frac{x^2}{(1+3x^2)^2}\,dx = \frac{\pi\sqrt{3}}{54} - \frac{1}{24}.$$ [8]

(b) Suppose that

$$E(\alpha) = \int_0^\infty \frac{\ln(1+\alpha^2 x^2)}{1+x^2}\,dx,$$

where α is a real parameter. Demonstrate that

$$\frac{dE}{d\alpha} = \frac{\pi}{1+\alpha}.$$ [8]

Solve this equation for $E(\alpha)$ and hence show that

$$\int_0^\infty \frac{\ln(1+x^2)}{1+x^2}\,dx = \pi \ln 2.$$ [4]

END OF PAPER