UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB & II (General)

Tuesday, 30 May, 2017 9:00 am to 12:00 pm

MATHEMATICS (1)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, ***3B****).*

Answers must be tied up in **separate** *bundles, marked* ***A, B or C*** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A ***separate*** *green master cover sheet listing all the questions attempted* ***must*** *also be completed.*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Green master cover sheet
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1B

(a) State the divergence theorem for a vector field $\boldsymbol{G}$. [2]

(b) Let A denote the *open* surface

$$x^2 + y^2 = 2z^2, \qquad 0 \leqslant z < h.$$

Sketch the surface A. [3]

(c) By applying the divergence theorem to a suitable *closed* surface, or otherwise, calculate

$$\int_A \boldsymbol{G} \cdot \mathrm{d}\boldsymbol{A},$$

where $\mathrm{d}\boldsymbol{A}$ is the unit area element pointing out of A, and

$$\boldsymbol{G} = \begin{pmatrix} x^3 + 2xy \\ y^3 + \sin x \\ z \end{pmatrix}.$$

[15]

2B

Consider the equation

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \frac{2v}{t+1},$$

where $v(x,t)$ is defined on $0 \leqslant x \leqslant \pi$ and is subject to the initial and boundary conditions

$$v(0,t) = 0, \;\; v(\pi,t) = f(t), \;\; v(x,0) = h(x),$$

for some functions $f(t)$ and $h(x)$.

(a) Using the substitution $v = (t+1)^2\, u$, show that u satisfies the diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\,,$$

and state the boundary and initial conditions satisfied by u. [5]

(b) Now consider the specific case when the functions f and h are given by

$$f(t) = 3(t+1)^2\,, \qquad h(x) = \frac{\sin{(2x)} + 3x}{\pi}\,.$$

Using the method of separation of variables, construct the solution $v(x,t)$. [13]

[*Hint: You may find it helpful to use the substitution* $u(x,t) = w(x,t) + \gamma\, x$, *for a suitably chosen constant* γ.]

(c) For $t \gg 1$, show that

$$v \sim \frac{3\,x\,t^2}{\pi}\,.$$

[2]

3B

An amplifier outputs a signal $x(t)$ given by the initial-value problem

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 2q\frac{\mathrm{d}x}{\mathrm{d}t} + \left(q^2+4\right)x = f(t)\,, \qquad x(0) = \frac{\mathrm{d}x}{\mathrm{d}t}(0) = 0\,, \qquad (\star)$$

for some constant $q > 0$ and input function $f(t)$.

(a) Show that the Green's function $G(t,\tau)$ for this problem is

$$G(t,\tau) = \begin{cases} 0 & 0 \leqslant t < \tau, \\ \frac{1}{2}e^{-q(t-\tau)}\sin\left[2(t-\tau)\right] & \tau \leqslant t\,. \end{cases}$$

[7]

Write down the general solution $x(t)$ of equation $(\star)$ in terms of an integral.

[2]

(b) Now consider the specific case $q = 0$ and

$$f(t) = \begin{cases} t_0 & 0 \leqslant t < t_0, \\ 0 & t_0 \leqslant t, \end{cases}$$

where $t_0 > 0$ is a constant. Calculate the solution of equation $(\star)$ in this case.

[8]

Find all values of t_0 for which $x(t) = 0$ for all $t \geqslant t_0$.

[3]

4A

(a) The Fourier transform of a function $f(t)$ is given by

$$\widetilde{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, \mathrm{d}t.$$

Write down the corresponding expression for the inverse Fourier transform.

[2]

(b) Consider the convolution of the functions f and g

$$h(z) = \int_{-\infty}^{\infty} f(t) g(z-t) \, \mathrm{d}t.$$

Prove that the Fourier transform of h is given by the product of the Fourier transforms of f and g.

[5]

(c) Find the Fourier transform of

$$f(\gamma, p, t) = \begin{cases} e^{-\gamma t} \sin pt & t > 0\,, \\ 0 & t \leqslant 0\,, \end{cases}$$

where $\gamma > 0$ and p are fixed parameters.

[*Hint: Write* $\sin pt$ *in terms of exponential functions.*]

[6]

(d) The current $I(t)$ flowing through a system is related to the applied voltage $V(t)$ by the equation

$$I(t) = \int_{-\infty}^{\infty} K(t-u) V(u) \, \mathrm{d}u,$$

where

$$K(\tau) = a_1 f(\gamma_1, p_1, \tau) + a_2 f(\gamma_2, p_2, \tau).$$

Here the function $f(\gamma, p, t)$ is as given in part (c), and all the $a_i, \gamma_i > 0$ and p_i are fixed parameters. By considering the Fourier transform of $I(t)$, find the relationship that must hold between a_1 and a_2 if the net charge Q, defined by

$$Q = \int_{-\infty}^{\infty} I(t') \, \mathrm{d}t'\,,$$

is to be zero for an arbitrary applied voltage.

[*Hint:* $\int_{-\infty}^{\infty} \exp[i\omega t'] \, \mathrm{d}t' = 2\pi\delta(\omega)$.]

[7]

5C

(a) When is an $n \times n$ matrix A diagonalisable? Give an example of a non-diagonalizable $n \times n$ matrix (for some n). What is a Hermitian matrix? Show that the eigenvalues of a Hermitian matrix are real, and that the corresponding eigenvectors are orthogonal. [5]

(b) Diagonalise the matrix

$$\mathsf{A} = \begin{pmatrix} 2 & -a & 0 \\ -a & 2 & 0 \\ 0 & 0 & c \end{pmatrix},$$

where $a > 0$ and $c > 0$ are real numbers and finds its eigenvectors. Sketch the surface [6]

$$\mathbf{x}^T \mathsf{A}\, \mathbf{x} = 1\,,$$

where $\mathbf{x} = (x, y, z)$, specifying the principal axes and, where appropriate, the semi-axis lengths. Note that different values of a may correspond to different surfaces. [9]

6C

(a) Let A and B be $n \times n$ Hermitian matrices, each with n distinct eigenvalues. Show that:

(i) the matrix $\mathsf{H} = i(\mathsf{AB} - \mathsf{BA})$ is Hermitian; [4]

(ii) the eigenvectors of A and B are identical if and only if $\mathsf{AB} = \mathsf{BA}$; [6]

(iii) if A and B commute, then the matrix $\mathsf{N} = \mathsf{A} + i\mathsf{B}$ is diagonalisable. [5]

(b) Suppose C is a unitary matrix, A is a Hermitian matrix, and p is a positive integer. Show that $(\mathsf{C}^{-1}\mathsf{AC})^p$ has real eigenvalues. [5]

7A

(a) Use the Cauchy-Riemann relations to show that, for any analytic function $f(x, y) = u(x, y) + i\,v(x, y)$, the relation $|\nabla u| = |\nabla v|$ must hold.

[2]

(b) Find the most general analytic function $f(z)$ of the variable $z = x + i\,y$ whose imaginary part is

$$(y \cos y + x \sin y) \exp x.$$

(*Your final expression for* $f(z)$ *should be in terms of* z, *not* x *and* y.)

[10]

(c) Find the radii of convergence of the following Taylor series:

(i)

$$\sum_{n=2}^{\infty} \frac{z^n}{\ln n}\,;$$

[3]

(ii)

$$\sum_{n=1}^{\infty} \left(\frac{n+p}{n} \right)^{n^2} z^n\,, \text{ with } p \text{ real.}$$

[5]

[*Hint: You may want to use the following result:*

$$a^n = e^{n \ln a}\,,$$

for some real a.]

8C

(a) Find the power series solution of the equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + \lambda\, y = 0\,, \qquad (\star)$$

where λ is a real parameter, about the point $x = 0$, and find suitable recurrence relations for the coefficients. For what values of λ does $(\star)$ have a polynomial solution? Find the solutions corresponding to two eigenvalues λ of your choice. [10]

(b) Consider the hypergeometric equation

$$x(1-x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + [\gamma - (1+\alpha+\beta)x]\frac{\mathrm{d}y}{\mathrm{d}x} - \alpha\,\beta\, y = 0\,,$$

where α, β and γ are real constants. Assuming a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n\, x^{n+\sigma} \quad \text{with} \quad a_0 \neq 0\,,$$

show that

$$\sigma = 0 \quad \text{or} \quad \sigma = 1 - \gamma\,,$$

and that

$$a_n = \frac{(n+\sigma+\alpha-1)(n+\sigma+\beta-1)}{(n+\sigma)(n+\sigma+\gamma-1)}\, a_{n-1}$$

for all $n \geqslant 1$. [10]

9B

(a) The Euler–Lagrange equation for extrema of the functional

$$D[y] = \int_a^b f(x, y, y')\,\mathrm{d}x\,,$$

where $y' = \mathrm{d}y/\mathrm{d}x$, is

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0. \qquad (\star)$$

If $f = f(y, y')$ does not depend explicitly on x show that $(\star)$ can be written as

$$\frac{\mathrm{d}h}{\mathrm{d}x} = 0\,.$$

for some h, which you should determine.

[3]

(b) A forest lies in the (x, y) plane. A new path through the forest is proposed, starting at $(x, y) = (-1, 1)$ and ending at $(x, y) = (1, 1)$. The density of undergrowth in the forest is given by $g(y)$, such that the total undergrowth D to be destroyed by the new path is

$$D = \int_{\mathcal{P}} g(y)\,\mathrm{d}s\,,$$

where $\mathrm{d}s$ is the arc-length element along the path $\mathcal{P}$.

(i) Given that the path always travels in the positive x direction, show that the path $y(x)$ that minimises the destruction of undergrowth satisfies

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{g}{\sqrt{1+y'^2}}\right) = 0\,.$$

[4]

(ii) In the specific case when $g = y^{-1}$, calculate the path $y(x)$. [9]

(iii) Sketch the path, and determine its length. [4]

10B

Consider the problem

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \epsilon\left[x\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \frac{\mathrm{d}u}{\mathrm{d}x} - u\right] = -\lambda u, \quad 0 \leqslant x \leqslant \pi, \quad u'(0) = u(\pi) = 0\,, \qquad (\star)$$

where $\epsilon \geqslant 0$ is a parameter, λ is a real constant, and $u' = \mathrm{d}u/\mathrm{d}x$. Express $(\star)$ in the form

$$\mathcal{L}u = \lambda u, \qquad (\star\star)$$

where $\mathcal{L}$ is an operator in Sturm–Liouville form.

[2]

Now consider the functional

$$I[v] = \int_0^\pi \left(p\,v'^2 + q\,v^2\right)\,\mathrm{d}x\,,$$

where $v(x)$ satisfies $v'(0) = v(\pi) = 0$, and is subject to the constraint

$$\int_0^\pi w\,v^2\,\mathrm{d}x = 1\,,$$

for smooth functions $p(x) > 0$, $q(x) \geqslant 0$ and $w(x) > 0$. Show that, for a particular choice of the functions p, q and w, which should be specified, finding extrema of I is equivalent to finding solutions of $(\star)$. Explain why the stationary values of I are the eigenvalues λ of equation $(\star\star)$. You may use the Euler–Lagrange equation without proof.

[8]

When $\epsilon = 0$, show that the smallest eigenvalue of $(\star\star)$ is $\lambda_0 = 1/4$, and the associated normalised eigenfunction is

$$U_0(x) = \sqrt{\frac{2}{\pi}}\cos\left(\frac{x}{2}\right).$$

Using $U_0(x)$ as a trial function, find an upper bound for the lowest eigenvalue λ of equation $(\star\star)$ when $\epsilon > 0$.

[10]

END OF PAPER