UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Wednesday, 14 June, 2017 9:00 am to 12:00 pm

MATHEMATICS (2)

Before you begin read these instructions carefully:

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

You may submit answers to ***all*** *of section A, and to no more than* ***five*** *questions from section B.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on one side of the paper only and begin each answer on a separate sheet. *(For this purpose, your section A attempts should be considered as one single answer.)*

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Tie up ***all of your section A answer*** *in a single bundle, with a completed blue cover sheet.*

Each section B question has a number and a letter (for example, ***11T****). Answers to these questions must be tied up in* ***separate*** *bundles, marked* ***R, S, T, W, X, Y*** *or* ***Z*** *according to the letter affixed to each question.* ***Do not join the bundles together****. For each bundle, a blue cover sheet* ***must*** *be completed and attached to the bundle, with the correct letter* ***R, S, T, W, X, Y*** *or* ***Z*** *written in the section box.*

A ***separate*** *green master cover sheet listing all the questions attempted* ***must*** *also be completed. (Your section A answer may be recorded just as A: there is no need to list each individual short question.)*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Green master cover sheet
Script paper

SPECIAL REQUIREMENTS
None.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1

Consider the two intersecting lines given by equations

$$\boldsymbol{r} = \begin{pmatrix}1\\0\\2\end{pmatrix} + s\begin{pmatrix}-1\\-1\\1\end{pmatrix}, \qquad \boldsymbol{r} = \begin{pmatrix}0\\-1\\0\end{pmatrix} + t\begin{pmatrix}1\\1\\2\end{pmatrix},$$

where s and t are real parameters.

(a) At what angle do the lines intersect? [1]

(b) Find the point at which they intersect. [1]

2

Consider $f(z) = z\mathrm{e}^{\mathrm{i}z}$, where $z = x + \mathrm{i}y$ and x and y are real.

(a) Find the real part of $f(z)$. [1]

(b) Find the imaginary part of $f(z)$. [1]

3

Consider the matrix

$$\begin{pmatrix}0 & a\\a & 0\end{pmatrix},$$

where $a \neq 0$ is a real number.

(a) Compute the matrix's eigenvalues. [1]

(b) Find its normalised eigenvectors. [1]

4

Find the first two non-zero terms in the Taylor series expansion of $x^3 \cos^2 x$ around the point $x = 0$. [2]

5

Find the first two non-zero terms in the Fourier series expansion of the function $\cos^4 x$, defined on $-\pi \leqslant x < \pi$. [2]

6

Consider the two vector fields

$$\boldsymbol{F} = (\sin x,\ \sin\ y, \sin\ z), \qquad \boldsymbol{G} = (\cos x,\ \cos\ y, \cos\ z).$$

(a) Calculate $\boldsymbol{F} \times \boldsymbol{G}$. [1]

(b) Hence find $\nabla \cdot (\boldsymbol{F} \times \boldsymbol{G})$. [1]

7

Consider the ordinary differential equation

$$\frac{d^2y}{dx^2} + 9y = -7\cos 4x.$$

(a) Calculate its complementary function. [1]

(b) Calculate its particular integral. [1]

8

If $\boldsymbol{F} = (y^2,\ x^2,\ 0)$, compute the surface integral

$$\int_S \boldsymbol{F} \cdot d\boldsymbol{S}$$

where

(a) S is a circular disk in the x-y plane centred on the origin with unit radius, and with surface normal pointing in the positive z-direction, [1]

(b) S is a square in the x-z plane centred on the origin with sides of unit length parallel to the x- and z-axes and with surface normal pointing in the positive y-direction. [1]

9

Consider the twice-differentiable function $u = u(\xi)$.

(a) If $\xi = x + 2\sqrt{y}$, calculate $\partial^2 u/\partial y^2$. [1]

(b) Show by substitution that $u(x + 2\sqrt{y})$ solves the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{2}\frac{\partial u}{\partial y} - y\frac{\partial^2 u}{\partial y^2} = 0\,.$$ [1]

10

A finite population of cockatiels has equal numbers of males and females. The probability that a male can sing is p. The probability that a female can sing is q.

(a) What is the probability that a cockatiel randomly selected from the population can sing? [1]

(b) A cockatiel is observed to sing. What is the probability that it is male? [1]

SECTION B

11T

(a) For the three position vectors **a**, **b** and **c** show explicitly using components ($\mathbf{a} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}$, etc.) that the vector triple product can be expressed as

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}\,. \qquad [5]$$

Hence, using properties of the scalar triple product (or otherwise), show that

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})\,. \qquad [5]$$

[*Note that* $\mathbf{a} \times \mathbf{b} \equiv \mathbf{a} \wedge \mathbf{b}$.]

(b) Write down the equation for a sphere S given that its centre is at position vector **a** and its radius is $p > 0$. [2]

Now suppose there is a second sphere S' with its centre at **b** and radius $q > 0$. What conditions must **a**, **b**, p and q satisfy in order for the two spheres S and S' to intersect in a circle? [4]

If S and S' do intersect, show that the plane in which the circle of intersection lies is given by

$$2(\mathbf{b} - \mathbf{a}) \cdot \mathbf{r} = p^2 - q^2 + b^2 - a^2\,,$$

where $a = |\mathbf{a}|$ and $b = |\mathbf{b}|$. [4]

12X

(a) (i) Write down the infinitesimally small volume element in spherical polar coordinates: (r, θ, ϕ). [2]

(ii) Assume that the Earth is a sphere of radius R and that the density ρ of the atmosphere varies with height h above the surface as $\rho = \rho_0 \exp(-h/h_0)$ where ρ_0 and h_0 are positive constants. Find an integral expression for the mass of the atmosphere and integrate to obtain an explicit formula in terms of ρ_0, h_0 and R. [8]

(b) (i) Sketch the region of integration for the following double integral:

$$\int_{x=-a}^{a} \int_{y=x^2}^{y=\sqrt{1-x^2}} dy\, dx\,,$$

where $a^2 = (\sqrt{5} - 1)/2$. [4]

(ii) Evaluate the integral, giving your answer in the form $A \sin^{-1} a + Ba^3$ where A and B are to be found. [6]

13Z

If a vector field can be written as the gradient of some scalar field, $\boldsymbol{F} = \nabla\Phi$, the vector field is said to be 'conservative'.

(a) Show, using Cartesian coordinates, that the line integral $\int_{\mathcal{C}} \boldsymbol{F} \cdot \boldsymbol{d}\mathbf{x}$ of a conservative vector field, $\boldsymbol{F}$, along some path, $\mathcal{C}$, can be calculated by using the scalar field evaluated at the end points. [2]

(b) Show, using Cartesian coordinates, that the curl of a conservative vector field is everywhere zero. [3]

(c) Calculate the curl of the vector field

$$\boldsymbol{F} = \left(2xy - z^3,\, x^2 - 2y,\, -3xz^2 - 1\right),$$

and thereby show that $\boldsymbol{F}$ is conservative. [3]

(d) Calculate the underlying scalar field Φ by evaluating the line integral of $\boldsymbol{F}$ along the piecewise linear path joining $(0, 0, 0)$ to $(x, 0, 0)$ to $(x, y, 0)$ to (x, y, z). Why is the result undefined with respect to an additive constant? [6]

(e) Calculate explicitly the line integral of $\boldsymbol{F}$ along the parabolic path described by (t, t, t^2) from $t = 1$ to $t = 2$. [6]

14R

(a) Suppose X is a discrete random variable taking positive integer values 0, 1, 2, … . Its probability distribution is denoted by $P(X)$. Write down expressions for the mean μ and variance σ^2. [2]

(b) When Cambridge United football team play a game, the probability that the total number of goals scored is X is given by

$$P(X) = A\frac{\lambda^X}{X!}\,, \qquad (\dagger)$$

where A is a normalisation constant and λ is a positive constant.

(i) If P is normalised, show that $A = \mathrm{e}^{-\lambda}$. [2]

(ii) In any game Cambridge United plays, what is the probability, in terms of λ, that K goals or fewer are scored? [1]

(iii) Show that the mean of the distribution P is λ. [5]

(c) In one season, the Cambridge United team play 10 games of football. You may assume that the probability of goal-scoring in every game is given by equation (†).

(i) What is the probability that at least one goal is scored in every game of the season? [2]

(ii) Show that the probability that only 1 goal is scored in total during the team's entire season is $10\lambda\mathrm{e}^{-10\lambda}$. [3]

(iii) Calculate the probability that 2 goals are scored in total during the team's season. [5]

15Y

(a) Find the relevant integrating factor and solve the following equations:

$$(i) \qquad (2xy^2 - y)dx + (2x - x^2y)dy = 0\,, \qquad [5]$$

$$(ii) \qquad (2y\sin x + 3y^4\,\sin x\,\cos x)dx - (4y^3\cos^2 x + \cos x)dy = 0\,. \qquad [5]$$

You may give these solutions in implicit form.

(b) Consider an equation of the form

$$y = p\,x + f(p)\,,$$

where $p \equiv \frac{dy}{dx}$ and f is a differentiable function. Show that

$$\big[x + f'(p)\big]\,\frac{dp}{dx} = 0$$

where $f'(p) \equiv \frac{df}{dp}$. [2]

Hence, or otherwise, find all solutions for the equation

$$y \;=\; p\,x \;+\; \frac{1}{p-1}\,. \qquad [8]$$

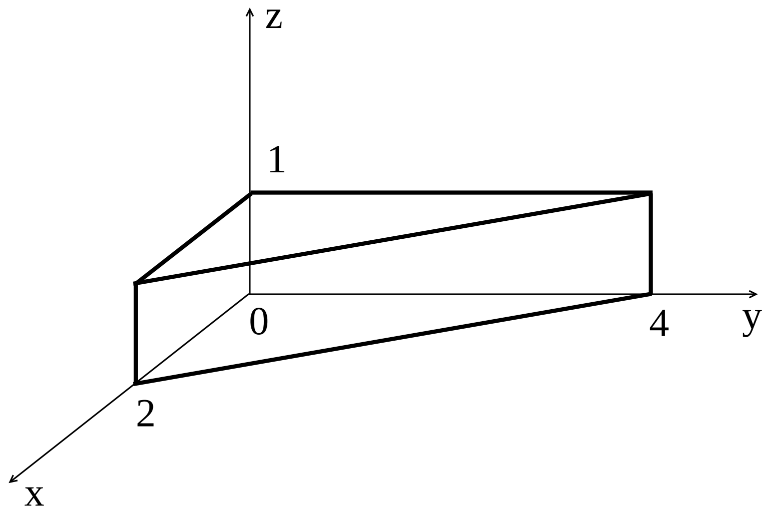
16X

(a) For the vector field $\boldsymbol{F}(x, y, z)$ give formulae in Cartesian coordinates for:

(i) $\nabla \cdot \boldsymbol{F}$, [1]

(ii) $\nabla \times \boldsymbol{F}$. [2]

(b) The closed surface S consists of the right triangular prism shown below.



For the vector field $\boldsymbol{F} = \left(0,\ (y + 2x - 4)^2,\ 1 - z^2\right)$:

(i) Calculate the outward flux for each of the five faces of the prism, and hence the total outward flux from S. [6]

(ii) Calculate $\nabla \cdot \boldsymbol{F}$. [3]

(iii) Find the volume integral of $\nabla \cdot \boldsymbol{F}$ over the interior of the prism. [6]

(iv) Comment on the relation between your answers to parts (b)(i) and (b)(iii). [2]

17Z

We can treat the following coupled system of differential equations as an eigenvalue problem:

$$2\frac{dy_1}{dt} = 2f_1 - 3y_1 + y_2,$$
$$2\frac{dy_2}{dt} = 2f_2 + y_1 - 3y_2,$$
$$\frac{dy_3}{dt} = f_3 - 4y_3,$$

where f_1, f_2 and f_3 is a set of time-dependent sources, and y_1, y_2 and y_3 is a set of time-dependent responses.

(a) If these equations are written using matrix notation,

$$\frac{d\mathbf{y}}{dt} + \mathbf{K}\mathbf{y} = \mathbf{f},$$

what are the elements of $\mathbf{K}$? Find the eigenvalues and eigenvectors of $\mathbf{K}$. [6]

(b) In the case when the system is not excited, $\mathbf{f} = \mathbf{0}$, find all of the solutions having the form

$$\mathbf{y}(t) = \mathbf{y}(0)e^{-\gamma t},$$

where $\gamma > 0$ is a decay constant. [4]

(c) If $\mathbf{f}$ is held constant at $\mathbf{f}_0$, the response vector $\mathbf{y}$ has the steady state value $\mathbf{y}_0$ (that is, with $\frac{d\mathbf{y}}{dt} = 0$). Write down $\mathbf{y}_0$ in terms of $\mathbf{f}_0$, and find $\mathbf{y}_0$ in the case where $\mathbf{f}_0 = (1, 1, 1)^T$. [6]

(d) Assume that $\mathbf{y}$ starts in the steady state solution $\mathbf{y}_0$ given in (c) with $\mathbf{f}_0 = (1, 1, 1)^T$. Now suppose the source function abruptly falls to zero, $\mathbf{f}_0 = (0, 0, 0)^T$, so that the response vector $\mathbf{y}$ moves away from $\mathbf{y}_0$. Writing $\mathbf{y}$ as a linear combination of the allowed solutions found in (b), derive an expression for the subsequent time evolution of the system. [4]

18S

(a) Suppose $f(x)$ is a 2π-periodic function defined on $-\pi \leqslant x < \pi$. Write down its Fourier series and give expressions for the coefficients appearing in it. Using the orthogonality relations or otherwise, determine the value of

$$\frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2 \, dx$$

in terms of the Fourier coefficients of f (Parseval's identity). [7]

(b) Show that the Fourier series of the 2π-periodic function $g(x) = x^3 - \pi^2 x$ for $-\pi \leqslant x < \pi$ is given by

$$g(x) = 12 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^p} \sin nx \,,$$

where the integer p should be determined. [7]

(c) Using Parseval's identity for g, show that

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945} \,.$$

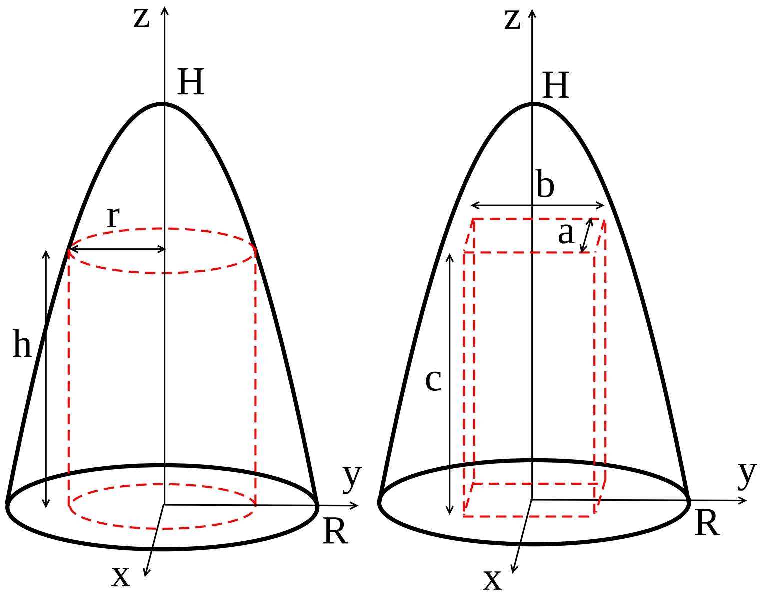
[6]

19W*

The interior region of a paraboloid of height H and radius R of the base is defined by the following inequalities

$$0 < z < H\left[1 - (x^2 + y^2)/R^2\right] .$$

Either a cylinder of height h and radius r or a rectangular parallelepiped with sides a, b and c can be inscribed into the paraboloid as shown by dashed lines in the left and right panels of the diagram, respectively.



By using the method of Lagrange multipliers,

(a) show that the maximum possible volume of a cylinder, V_{c}, inscribed into the paraboloid as shown in the diagram above is

$$V_{\mathrm{c}} = \frac{\pi R^2 H}{4} ,$$ [7]

(b) find in terms of H and R the maximum possible volume of the rectangular parallelepiped, V_{p}, inscribed into the paraboloid, [11]

(c) and thus determine which shape can produce a larger volume. [2]

[*Hint: You need not prove that suitable extrema you find are actually maxima.*]

20Y*

(a) (i) Solve the equation

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{1+x}\,,$$

subject to the boundary condition $y(0) = 1$. [4]

(ii) Solve the equation

$$\frac{dy}{dx} + \tfrac{1}{3}y = e^{x}y^{4}\,,$$

subject to the boundary condition $y(0) = 1$. [5]

(b) The following partial differential equation on the given interval,

$$\frac{\partial u}{\partial t} + u = \frac{\partial^2 u}{\partial x^2}\,, \qquad 0 \leqslant x \leqslant L\,, \quad t \geqslant 0\,, \qquad (\ddagger)$$

has the boundary conditions $u(0,t) = u(L,t) = 0$. By using the separable function $u(x,t) = X(x)T(t)$, show that the equation (‡) may be written as

$$\frac{1}{T}\frac{dT}{dt} + 1 = \frac{1}{X}\frac{d^2X}{dx^2} = -k^2\,,$$

with k a constant.

Determine the functions $X(x), T(t)$ satisfying the boundary conditions.

Hence, write down the general solution of the partial differential equation (‡). [11]

END OF PAPER