UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Monday, 12 June, 2017 9:00 am to 12:00 pm

MATHEMATICS (1)

Before you begin read these instructions carefully:

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

You may submit answers to ***all*** *of section A, and to no more than* ***five*** *questions from section B.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on one side of the paper only and begin each answer on a separate sheet. *(For this purpose, your section A attempts should be considered as one single answer.)*

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Tie up ***all of your section A answer*** *in a single bundle, with a completed blue cover sheet.*

Each section B question has a number and a letter (for example, ***11T****). Answers to these questions must be tied up in* ***separate*** *bundles, marked* ***R, S, T, W, X, Y*** *or* ***Z*** *according to the letter affixed to each question.* ***Do not join the bundles together****. For each bundle, a blue cover sheet* ***must*** *be completed and attached to the bundle, with the correct letter* ***R, S, T, W, X, Y*** *or* ***Z*** *written in the section box.*

A ***separate*** *green master cover sheet listing all the questions attempted* ***must*** *also be completed. (Your section A answer may be recorded just as A: there is no need to list each individual short question.)*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Green master cover sheet
Script paper

SPECIAL REQUIREMENTS
None.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1

(a) Calculate

$$\int_{-2}^{-1} \frac{1}{x}\, dx\,.$$

[1]

(b) Given $\frac{dy}{dx} = -y$ and $y(0) = 2$, calculate $y(2)$. [1]

2

(a) Calculate $\int \sin 2x \cos x \, dx\,.$ [1]

(b) Calculate $\int x \ln x \, dx$ for $x > 0\,.$ [1]

3

(a) Differentiate $y = 2^x$ with respect to x. [1]

(b) For $y = \tan t$ and $x = \cot t$, calculate $\frac{dy}{dx}$ in terms of t. [1]

4

(a) For the curve $y^3 + x^2 = 2$ calculate $\frac{dy}{dx}$ at $x = 1$. [1]

(b) Hence calculate the equation of the tangent line at the same point. [1]

5

(a) Verify that the polynomial $2x^3 + 3x^2 - 11x - 6$ changes sign between $x = 0$ and $x = 3$ and so find one of its roots by trial. [1]

(b) Hence factorise the polynomial. [1]

6

(a) Write the polynomial $1 + 8x + 24x^2 + 32x^3 + 16x^4$ in the form $(a + bx)^m\,.$ [1]

(b) Hence sketch the graph of the polynomial in the range $-1 \leqslant x \leqslant 1$ and calculate the points of intersection with the x and y axes. [1]

7

(a) Write $\sin^2\left(\frac{\pi}{12}\right)$ in terms of $\sin\left(\frac{\pi}{6}\right)$. [1]

(b) Given $\sqrt{3} \simeq 1.732$, calculate $\sin^2\left(\frac{\pi}{12}\right)$ to two decimal places. [1]

8

(a) Solve $\cos x = \sin 2x$ for $0 \leqslant x \leqslant \frac{\pi}{2}$. [1]

(b) Solve $1 + \cos 2x = \cos^2 x$ for $\pi < x < 2\pi$. [1]

9

(a) Find the general solution of $\frac{dy}{dx} = y^2$ for $y \neq 0$. [1]

(b) Find the particular solution which satisfies $y(1) = 1$. [1]

10

Three points are $O = (0, 0)$, $\quad A = (3, 4)$, $\quad B = (5, 12)$.

(a) Calculate the lengths OA and OB. [1]

(b) Calculate the cosine of the angle AOB. [1]

SECTION B

11T

(a) Express the complex number $z = x + iy$ in terms of the plane polar coordinates (r, θ) in the Argand plane. [2]

(b) Find in terms of x and y the real and imaginary parts for the following functions:

(i) $\ln z$,

(ii) $1/z$,

(iii) $\bar{z}(z^2 - |z|^2)$,

(iv) $\cosh z$,

where $\bar{z} \equiv z^*$ is the complex conjugate of z. [6]

(c) Sketch contours in the Argand plane along which the real part of the function (b)(i) is constant. Similarly sketch contours along which the real part of the function (b)(ii) is constant. [6]

(d) Find all solutions of the equation $\cosh z = -2$. [6]

12W

(a) An implicit equation for any of the real variables, x, y and z in terms of the other two can be written as $F(x, y, z) = 0$, where F is some function.

(i) For a general function F, derive the reciprocity relation,

$$\left(\frac{\partial y}{\partial x}\right)_z \left(\frac{\partial x}{\partial y}\right)_z = 1 \ ,$$

[3]

and cyclic relation,

$$\left(\frac{\partial y}{\partial x}\right)_z \left(\frac{\partial x}{\partial z}\right)_y \left(\frac{\partial z}{\partial y}\right)_x = -1 \ ,$$

[3]

clearly stating any assumption(s) you are making. [1]

(ii) For

$$F = xyz - \sinh(x + z) \ ,$$

find $\left(\frac{\partial x}{\partial y}\right)_z$, $\left(\frac{\partial x}{\partial z}\right)_y$, $\left(\frac{\partial y}{\partial z}\right)_x$. [3]

(b) Consider a differential form $P(x, y)dx + Q(x, y)dy$ with

$$P(x, y) = -\frac{aby}{a^2x^2 + b^2y^2} \quad \text{and} \quad Q(x, y) = \frac{abx}{a^2x^2 + b^2y^2} \ ,$$

(defined for $(x, y) \neq (0, 0)$), where a and b are real non-zero parameters.

(i) Find all values of a and b for which this differential form is exact and thus can be written as $df = P(x, y)dx + Q(x, y)dy$. [3]

(ii) Find $f(x, y)$. [5]

(iii) Using the expression for $f(x, y)$ found in (b)(ii) above, find the general solution of the differential equation $df = 0$, giving your answer as an explicit function $y(x)$. [2]

13Y

(a) Find the general solution for the equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 4x\,.$$

[4]

(b) Derive the general form of the solution for the equation

$$\frac{dy}{dx} + P(x)\,y = Q(x),$$

where P and Q are arbitrary functions. [6]

(c) Solve the equation

$$(x^2+1)\frac{dy}{dx} + 3x^3y = 6xe^{-3x^2/2}\,,$$

subject to the boundary condition $y(0) = 1$. [10]

14W

(a) Assume that the first and second order partial derivatives of the function $f(x, y)$ at the point (x_0, y_0) exist and also at this point that

$$\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \neq 0\,.$$

(i) Give the condition in terms of ∇f for the point (x_0, y_0) to be stationary. [1]

(ii) Give the conditions on the second-order derivatives used for the classification of stationary points. [3]

(b) (i) Find and classify all stationary points of

$$f(x, y) = \exp\left(\frac{x^2+y^2}{2} + \frac{ax^4}{4}\right)\,,$$

where a is a parameter in the range $-\infty < a < \infty$. [8]

(ii) For $a = 1$ and $a = -1$, sketch the contour lines of $f(x, y)$ in the region $-2 < x < 2$, $-2 < y < 2$. [4]

(iii) For the case $a = -1$, find explicit equations for the contour lines passing through the stationary points (if any). [4]

15S

(a) Give the Taylor series expansion of a function $f(x)$ about a point $x = a$ keeping the first n terms, and give an expression for the remainder term R_n. [4]

(b) Taking $f(x) = x^{1/3}$, find an approximation for $9^{1/3}$ as a sum of fractions using the first three terms of the Taylor series expansion of $f(x)$ about $x = 8$. By considering the expression for the remainder term in this case, show that the absolute value of the error is less than $1/4000$. [8]

(c) Find, by any method, the Taylor series expansion about $x = 0$, up to and including the term in x^4, of the following functions:

(i) $$e^{\sin x}\,,$$ [4]

(ii) $$\frac{e^{-x^3}}{\cosh x}\,.$$ [4]

16R

(a) A standard pack of 52 playing cards contains 13 kinds of card in four different suits. The pack is shuffled and five cards drawn. What is the probability of drawing 'four of a kind', i.e. the same kind of card from each of the four suits, plus any extra card. Leave your answer in fractional form $N/\binom{52}{5}$, where N is to be determined. [4]

[*Note the equivalent notation:* $\begin{pmatrix} n \\ m \end{pmatrix} \equiv {}^nC_m$.]

(b) The probability that a violist plays a wrong note is p. A piece of viola music contains 100 notes. What is the probability that n notes are wrong in a performance of the piece by the violist? [2]

If more than half the notes are wrong, the audience will start booing. What is the probability of that happening? You may give your answer in the form of a sum. [3]

(c) Let X be a continuous random variable that takes any real value. It is described by the logistic probability distribution

$$f(x) = \frac{1}{4s}\mathrm{sech}^2\left(\frac{x-\mu}{2s}\right),$$

where μ and s are parameters ($s \neq 0$).

(i) Verify that f is normalised. [3]

(ii) Calculate the probability that $0 < X < \mu$. [3]

(iii) Show that the mean of the distribution is μ. [5]

[*Hint:* $d(\tanh z)/dz = \mathrm{sech}^2 z$.]

17T

(a) Evaluate the following indefinite integrals:

$$(i) \qquad \int \frac{dx}{x \ln x}\,, \qquad (x > 1)\,, \qquad [4]$$

$$(ii) \qquad \int \frac{\sinh^3 x}{\cosh^2 x} dx\,. \qquad [7]$$

(b) Define the definite integral

$$I_n \equiv \int_{-\infty}^{\infty} x^{2n}\, e^{-x^2}\, dx\,,$$

where n is an integer ($n \geqslant 0$). Derive the recursion relation

$$I_{n+1} = \left(n + \tfrac{1}{2}\right)\, I_n\,.$$

Hence evaluate I_2 and I_3. [9]

[*You may assume that* $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}\,.$]

18Z

Consider the 2 by 3 matrix

$$\mathbf{A} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1+\sqrt{3} & 1 & 1-\sqrt{3} \\ 1-\sqrt{3} & 1 & 1+\sqrt{3} \end{pmatrix}.$$

(a) Calculate the matrices $\mathbf{A}\mathbf{A}^T$ and $\mathbf{A}^T\mathbf{A}$. [4]

(b) Calculate the eigenvalues, λ_n, and normalised eigenvectors, $\mathbf{u}_n$, of $\mathbf{A}\mathbf{A}^T$. [4]

(c) Calculate the eigenvalues, σ_m, and normalised eigenvectors, $\mathbf{v}_m$, of $\mathbf{A}^T\mathbf{A}$. [9]

(d) Calculate $\mathbf{A}\mathbf{v}_m$ for each of $m = 1, 2, 3$, and compare with $\mathbf{u}_n$. [3]

19S*

(a) Using the Newton-Raphson method, state the recursion relation for finding an approximate root of the function $f(x) = x^2 - a$, where $a > 0$. [3]

(b) Let $x_* = \sqrt{a}$ be the exact solution, and let $e_n = x_n - \sqrt{a}$ be the error at the n^{th} iteration. Show that

$$e_{n+1} = \frac{e_n^2}{2x_n},$$

and deduce that if $x_0 > x_*$, then $x_n > x_*$ for all n. [3]

(c) Let $a = 2$, and assume that we know that the exact solution, x_*, of $f(x) = x^2 - 2 = 0$ satisfies $1.4 < x_* < 1.5$. From part (b), deduce that with $x_0 = 1.5$, after three iterations, x_3 approximates x_* with accuracy 10^{-11}. [6]

(d) Determine whether the following series are convergent:

(i) $$\sum_{n=2}^{\infty} \frac{1}{n \ln n},$$ [4]

(ii) $$\sum_{n=2}^{\infty} \frac{1}{n \ln^2 n}.$$ [4]

20Y*

(a) Find the solution for

$$\frac{dy}{dx} = \frac{2x^3 + y^3}{3xy^2}.$$

[8]

(b) Sketch the family of curves

$$y = cx^2 \qquad (\dagger)$$

for integer values of c, $-3 \leqslant c \leqslant 3$. Find the family of curves that are orthogonal to (†); sketch one example of these orthogonal curves. [12]

END OF PAPER