UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS **Part IB & II (General)**

Tuesday, 24 May, 2016 9:00 am to 12:00 pm

MATHEMATICS (1)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, ***6A****).*

Answers must be tied up in **separate** *bundles, marked* ***A, B or C*** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A ***separate*** *green master cover sheet listing all the questions attempted* ***must*** *also be completed.*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
3 blue cover sheets and treasury tags
Green master cover sheet
Script paper

SPECIAL REQUIREMENTS
Calculator - students are permitted to bring an approved calculator.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1B

(a) State the divergence theorem for a vector field $\mathbf{F}(x, y, z)$. [2]

(b) Let the surface S be defined as $S = S_1 \cup S_2 \cup S_3$, where

$$\begin{aligned} S_1 &= \{(x, y, z) : x^2 + y^2 = 2 - z,\ 1 \leqslant z \leqslant 2\}, \\ S_2 &= \{(x, y, z) : x^2 + y^2 = 1,\ 0 \leqslant z \leqslant 1\}, \\ S_3 &= \{(x, y, z) : z = 0,\ x^2 + y^2 \leqslant 1\}. \end{aligned}$$

Sketch all four surfaces. [4]

(c) Given that $\mathbf{F}(x, y, z) = (2xy + x^6,\ -y^2 + y^4,\ z)$, find $\oint_S \mathbf{F} \cdot d\mathbf{S}$, where $d\mathbf{S}$ is an element of vector area pointing in the direction of the outward normal to S. [14]

2B

A string of uniform density per unit length ρ is stretched under tension along the x-axis and undergoes small transverse oscillations in the (x, y) plane with amplitude $y(x, t)$. The waves in the string travel with velocity c and the equation of motion satisfied by $y(x, t)$ is

$$\frac{1}{c^2} \frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}.$$

(a) If the string is fixed at $x = 0$ and $x = L$, derive the general separable solution for the amplitude $y(x, t)$. [9]

(b) For $t < 0$ the string is at rest. At $t = 0$ the string is struck by a hammer in the interval $[\ell - a/2, \ell + a/2]$, the distance being measured from one end. The effect of the hammer is to impart a constant velocity v to the string inside the interval and zero velocity outside the interval. Calculate the proportion of the total energy given to the string in each mode. [*You may assume the kinetic energy formula* $K.E. = \int_0^L \frac{\rho}{2} (\partial y / \partial t)^2 dx$.] [9]

(c) If $\ell = L/5$ and $a = L/7$, identify all the modes of the string which are not excited by the hammer. [2]

3B

Consider the linear differential operator $\mathcal{L}$ defined by

$$\mathcal{L}y \equiv -\frac{d^2y}{dx^2} + y$$

on the interval $0 \leqslant x < \infty$. The boundary conditions are given by $y(0) = 0$ and $\lim_{x\to\infty} y(x) = 0$.

(a) Find the Green's function $G(x,\xi)$ for $\mathcal{L}$ satisfying these boundary conditions. Hence, or otherwise, obtain the solution of

$$\mathcal{L}y = \begin{cases} 1, & 0 \leqslant x \leqslant \mu \\ 0, & \mu < x < \infty \end{cases},$$

subject to the above boundary conditions, where μ is a positive constant. [14]

(b) Show that your piecewise solution is continuous at $x = \mu$ and has the value

$$y(\mu) = \frac{1}{2}(1 + e^{-2\mu} - 2e^{-\mu}).$$

[6]

4A

The waveform $\phi(t)$ transmitted by an analogue radio is produced by modulating a carrier wave $c(t) = \cos(\Omega t)$ of frequency Ω by the signal $s(t)$ to be broadcast such that

$$\phi(t) = (1 + s(t))c(t).$$

The Fourier transform of $s(t)$ is $\tilde{s}(\omega)$ and the maximum amplitude of $s(t)$ does not exceed unity.

(a) What is the Fourier transform $\tilde{c}(\omega)$ of the carrier wave? What is $\tilde{\phi}(\omega)$, the Fourier transform of $\phi(t)$, in terms of $\tilde{s}(\omega)$? [4]

(b) For the case

$$s(t) = \frac{1 - \cos t}{t^2},$$

compute the Fourier transform to show that

$$\tilde{s}(\omega) = \begin{cases} \pi(1 - |\omega|) & |\omega| < 1\,, \\ 0 & |\omega| \geqslant 1\,. \end{cases}$$

Sketch $|\tilde{s}(\omega)|$ and hence $|\tilde{\phi}(\omega)|$ for $\Omega > 1$. [*Hint: The Fourier transform of* $1/t^2$ *is* $-\pi|\omega|$.] [6]

(c) To reduce the bandwidth requirements for the radio, a 'single side-band' design was adopted such that the new transmitted signal $\rho(t)$ has a Fourier transform given by

$$\tilde{\rho}(\omega) = \begin{cases} \tilde{\phi}(\omega) & |\omega| < \Omega\,, \\ 0 & |\omega| \geqslant \Omega\,. \end{cases}$$

For the case $\Omega = \frac{3}{4}$ and using the form of $\tilde{\phi}(\omega)$ determined in (b), sketch $|\tilde{\phi}(\omega)|$ and $|\tilde{\rho}(\omega)|$. Determine $\rho(t)$. [10]

5C

(a) Given an $n \times n$ matrix $\mathbf{M}$ and the identity $\mathbf{I}$, show that the matrices $\mathbf{I} + \mathbf{M}$ and $(\mathbf{I} - \mathbf{M})^{-1}$ commute. [4]

(b) Show that the three eigenvalues of a real orthogonal 3×3 matrix are $e^{+i\alpha}$, $e^{-i\alpha}$ and $+1$ or -1, where α is real. [4]

(c) For a real antisymmetric matrix $\mathbf{A}$, the matrix $\mathbf{N}$ is defined by

$$\mathbf{N} = (\mathbf{I} + \mathbf{A})(\mathbf{I} - \mathbf{A})^{-1} .$$

(i) Show that $\mathbf{N}$ is orthogonal. [4]

(ii) Show that eigenvectors of $\mathbf{A}$ are also eigenvectors of $\mathbf{N}$. What is the relation between the eigenvalues of $\mathbf{A}$ and the eigenvalues of $\mathbf{N}$? [3]

(iii) Show that when $\mathbf{A}$ and $\mathbf{N}$ are 3×3 matrices, $\det \mathbf{N} = 1$ and that there exists a direction $\mathbf{x}$ in which $\mathbf{A}\mathbf{x} = 0$, with $\mathbf{x} \neq 0$. [5]

6C

(a) What does it mean for an $n \times n$ square matrix to be diagonalisable? [2]

(b) Suppose that $\mathbf{A}$ is a complex $n \times n$ matrix such that $\mathbf{A}^p = \mathbf{0}$ for some positive integer p. Show that $\mathbf{A}$ has 0 as an eigenvalue. Show that $\mathbf{A}$ is not diagonalisable unless $\mathbf{A} = \mathbf{0}$. [6]

(c) Let $\mathbf{B}$ and $\mathbf{C}$ be the matrices

$$\mathbf{B} = \begin{pmatrix} 4+2\alpha & -2 & -2-4\alpha \\ 3\alpha & -3 & 9-6\alpha \\ 2+\alpha & -1 & -1-2\alpha \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 0 & 2 & 6 \\ 3 & 3 & 3 \\ 3 & 1 & -3 \end{pmatrix} .$$

By considering the characteristic polynomials of $\mathbf{B}$ and $\mathbf{C}$, determine whether $\mathbf{B}$ and $\mathbf{C}$ are diagonalisable. [12]

7A

Consider the mapping $z = f(\zeta)$ such that $G(z) = G(f(\zeta)) = \psi(\zeta)$, where f, G, ψ are complex functions and z, ζ are complex variables.

(a) What condition(s) must be satisfied for $\psi(\zeta)$ to be analytic? [3]

(b) Suppose that $\psi(\zeta) = \ln(\zeta + 2)$ and $f(\zeta)$ is defined by

$$\frac{df}{d\zeta} = \frac{i}{\sqrt{(\zeta+1)(\zeta-1)}}, \qquad (\star)$$

where $\zeta = 0$ maps to $z = 0$.

(i) By integrating $(\star)$, show that the upper half of the ζ plane maps onto the region R defined by $|\mathrm{Re}(z)| \leqslant \frac{1}{2}\pi$, $\mathrm{Im}(z) \geqslant 0$. Determine the location of any points in the region R where $G(z)$ is not analytic. How do these relate to points in the ζ plane? [*Hint:* $\sin(x + iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$.] [7]

(ii) The vector field $\mathbf{u} = (u, v)$ in the ζ plane is given by $u - iv = d\psi/d\zeta$. How does the magnitude of $\mathbf{u}$ vary across the upper half of the ζ plane? In what direction is $\mathbf{u}$ oriented? [3]

(iii) The vector field $\mathbf{U} = (U, V)$ is defined in the region R of the z plane by $U - iV = dG/dz$. Determine this field and use a sketch to illustrate the orientation of the vector field in this region. [7]

8C

(a) Define the terms *ordinary point* and *regular singular point* for a second order linear differential equation of the form

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0\,,$$

and explain briefly the reason for distinguishing between them. [4]

(b) Let $f(x)$ and $g(x)$ be two differentiable functions on $x \in [a,b]$. Define the Wronskian $W(f,g)(x)$ and show that if $W(f,g)(x_0) \neq 0$ for $x_0 \in [a,b]$ then f and g are linearly independent on $[a,b]$. [6]

(c) Find power series solutions of the equation

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + k^2 y = 0\,,$$

about the point $x = 0$, giving the recurrence relation for the coefficients. Determine the radius of convergence of the solutions about $x = 0$. [10]

9B

(a) Suppose that the speed of light $c(y)$ varies continuously through a medium and is a function of the distance from the boundary $y = 0$. Use Fermat's principle to show that the path $y(x)$ of the light ray is given by the solution of

$$c(y)y'' + c'(y)(1 + y'^2) = 0.$$

[4]

(b) The curve assumed by a uniform chain, which is suspended between two points $(-a, b)$ and (a, b) minimises the potential energy,

$$\int_{-a}^{a} y(1 + y'^2)^{\frac{1}{2}} dx,$$

subject to the constraint that its length remains fixed,

$$\int_{-a}^{a} (1 + y'^2)^{\frac{1}{2}} dx = 2L,$$

where $L > a$.

(i) Show that the curve is the catenary

$$y - y_0 = k \cosh\left(\frac{x - x_0}{k}\right),$$

where k, x_0 and y_0 are constants. [8]

(ii) Find an equation for k and show, using a graphical method, that it has a unique positive solution. [8]

10B

(a) Let E_n be the eigenvalues of the self-adjoint operator H, and ψ_n be the corresponding orthonormal eigenfunctions. Let $F[\phi]$ be the functional

$$F[\phi] = \frac{\int \phi^* H \phi \, d\tau}{\int \phi^* \phi \, d\tau},$$

(for finite, non-zero $\int \phi^* \phi \, d\tau$) where $\phi(\tau)$ is a finite arbitrary function and the integration extends from $-\infty$ to $+\infty$.

(i) If ψ_n is an eigenfunction of H, show that

$$F[\psi_n] = E_n.$$

[2]

(ii) Show that if $\phi = \psi_n + \delta\phi$, where $\delta\phi$ is an arbitrary infinitesimal variation, the functional $F[\psi_n]$ is stationary and that

$$(H - F[\psi_n])\phi = 0.$$

State any assumptions made. [6]

(iii) Show that $F[\phi]$ gives an upper bound to the exact ground state eigenvalue E_0 by expanding ϕ as

$$\phi = \sum_n a_n \psi_n,$$

where $\int \phi^* \phi \, d\tau = \sum_n |a_n|^2$. [2]

(b) Consider a particle of mass m moving in one dimension. For the operator $H = \left(-\hbar^2/2m\right) \frac{d^2}{dx^2}$ for $-a < x < a$, with $H = 0$ elsewhere, the exact ground state eigenvalue E_0 is given by

$$E_0 = \frac{6\hbar^2}{5ma^2},$$

where a and $\hbar$ are constants. Use the Rayleigh-Ritz method to obtain the lowest upper bound of the value of E_0 by choosing a trial function

$$\phi_{\text{trial}}(\lambda, x) = \begin{cases} (a^2 - x^2)(1 + \lambda x^2), & -a \leqslant x \leqslant a \\ 0, & |x| > a \end{cases},$$

where λ is a real variational parameter. [*You may assume that the roots to the quadratic equation* $13a^4\lambda^2 + 98a^2\lambda + 21$ *are approximately* $-1/5a^2$ *and* $-7/a^2$.] [10]

END OF PAPER