UNIVERSITY OF CAMBRIDGE

## NATURAL SCIENCES TRIPOS Part IA

Wednesday, 8 June, 2016 9:00 am to 12:00 pm

# MATHEMATICS (2)

## Before you begin read these instructions carefully:

*The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.*

*You may submit answers to* ***all*** *of section A, and to no more than* ***five*** *questions from section B.*

*The approximate number of marks allocated to a part of a question is indicated in the right hand margin.*

***Write on one side of the paper only and begin each answer on a separate sheet.*** *(For this purpose, your section A attempts should be considered as one single answer.)*

*Questions marked with an asterisk (*) require a knowledge of B course material.*

## After the end of the examination:

*Tie up* ***all of your section A answer*** *in a single bundle, with a completed blue cover sheet.*

*Each section B question has a number and a letter (for example,* ***11S****). Answers to each question must be tied up in* ***separate*** *bundles and marked (for example,* ***11S, 12W*** *etc) according to the number and letter affixed to each question.* ***Do not join the bundles together****. For each bundle, a blue cover sheet* ***must*** *be completed and attached to the bundle, with the correct number and letter written in the box.*

*A* ***separate*** *green master cover sheet listing all the questions attempted* ***must*** *also be completed. (Your section A answer may be recorded just as A: there is no need to list each individual short question.)*

***Every cover sheet must bear your examination number and desk number.***
***Calculators are not permitted in this examination.***

***STATIONERY REQUIREMENTS***
*6 blue cover sheets and treasury tags*
*Green master cover sheet*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

## SECTION A

**1**

Let $z = x + iy$ be a complex number with $x$ and $y$ real valued. Find all the solutions of $|z-1| = (z-1)$ in terms of $x$ and $y$. [2]

**2**

Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be a position vector in a Cartesian basis. Find the line of intersection of the two planes

$$\frac{x-1}{4} = \frac{y-2}{3} \quad \text{and} \quad \frac{y-2}{3} = \frac{z+1}{2},$$

writing your answer in the form $\mathbf{r} = \mathbf{r_0} + \lambda\mathbf{t}$ where $\mathbf{r_0}$ is a constant vector, $\lambda$ is a real parameter and the vector, $\mathbf{t}$, has unit length. [2]

**3**

Let $(x, y)$ and $(r, \theta)$ be coordinates on the real two-dimensional plane. The coordinates are related by the transformation,

$$x = r\cos 2\theta, \qquad y = r\sin 2\theta,$$

with $0 \leqslant \theta < \pi$ and $r > 0$. Writing the function $f$ either as $f(x, y)$ or as $f(r, \theta)$ find $\left(\frac{\partial f}{\partial r}\right)_\theta$, $\left(\frac{\partial f}{\partial \theta}\right)_r$ in terms of $\left(\frac{\partial f}{\partial y}\right)_x$ and $\left(\frac{\partial f}{\partial x}\right)_y$. [2]

**4**

Two players, Alice and Bob, play a game with a pair of fair dice. Each player, in turn, throws both dice together. Alice throws first. The winner is the first player to throw two sixes in one throw.

(a) What is the probability that Alice wins on her second turn to throw? [1]

(b) What is the probability that Alice wins eventually? Give your answer in terms of a single fraction. [1]

**5**

Determine which of the following matrices can be written in real diagonal form by a suitable choice of basis. In each case justify your answer.

(a) $\mathbf{A} = \begin{pmatrix} 4 & 4 \\ 4 & 3 \end{pmatrix}$. [1]

(b) $\mathbf{B} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. [1]

**6**

Consider the function $\phi(x, y, z) = xye^{xz}$. Let $\mathbf{F} = \nabla\phi$.

(a) Evaluate $\nabla \times \mathbf{F}$. [1]

(b) Evaluate the line integral $\int \mathbf{F} \cdot \mathbf{dr}$ along the curve with Cartesian coordinates $(x, y, z)$ given parametrically by $x = \sin t$, $y = \cos 2t$ and $z = 0$ from $t = 0$ to $t = \dfrac{\pi}{8}$. [1]

**7**

(a) Find the Fourier series for the function $f(x) = \sin x + 2\sin x\cos x + \sin 3x$ on the interval $-\pi \leqslant x \leqslant \pi$. [1]

(b) Find the Fourier series for the derivative of $f(x)$. [1]

**8**

Solve the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$$

subject to the boundary conditions, $y = 0$ when $x = 0$ and $y = \pi e^{\pi}$ when $x = \pi$. [2]

**9**

Consider two concentric spherical shells with mass density, $\rho(r)$, given in spherical polar coordinates, where $r$ is the radius, and $a$, $b$, $c$ and $d$ are positive constants with $a < b < c < d$.

$$\rho(r) = \begin{cases} 1, & \text{if } a \leqslant r \leqslant b, \\ \dfrac{1}{r^4}, & \text{if } c \leqslant r \leqslant d, \\ 0, & \text{otherwise.} \end{cases}$$

Find the total mass. [2]

**10**

(a) Determine the Taylor series of the function $\phi(x, y) = xy + x^4 + x^2y^2 + y^3x^2 + x^5 + y^5$ about the origin up to and including terms of order 3. [1]

(b) What type of stationary point does $\phi(x, y)$ have at the origin? [1]

## SECTION B

**11S**

Three non-zero vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly independent with $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \neq 0$. Assume they are oriented such that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) > 0$.

(a) (i) Find an expression, not in parametric form, which the vector $\mathbf{r}$ must satisfy if it lies in the plane containing the origin, $\mathbf{0}$, and the points with position vectors $\mathbf{a}$ and $\mathbf{b}$.

(ii) Now suppose instead that the vector $\mathbf{r}$ lies above or below this plane and state a condition that ensures it lies on the same side of the plane as $\mathbf{c}$. [5]

(b) Show that the equation of the plane through the points with position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ can be expressed as

$$\mathbf{r} \cdot (\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

[5]

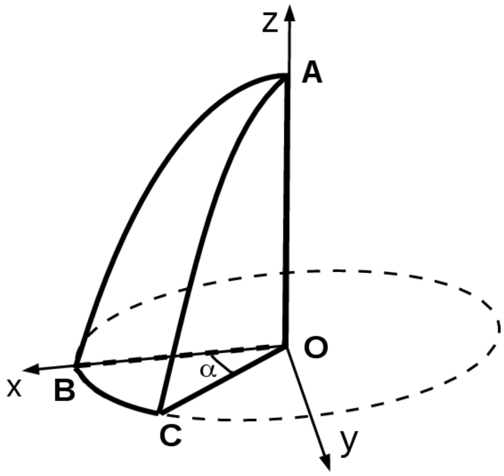
(c) Find necessary and sufficient conditions for the point with position vector $\mathbf{r}$ to lie inside, or on, the tetrahedron formed by the vertices $\mathbf{0}$, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. [5]

(d) Suppose the vectors are explicitly $\mathbf{a} = (0,\ -2,\ 1)$, $\mathbf{b} = (2,\ 2,\ 0)$, and $\mathbf{c} = (-1,\ 1,\ 2)$. Find the perpendicular distance between $\mathbf{c}$ and the opposite face of the tetrahedron given in (c). [5]

[Note: $\mathbf{a} \times \mathbf{b}$ and $\mathbf{a} \wedge \mathbf{b}$ are alternative notations for the cross product, or vector product, of the two vectors $\mathbf{a}$ and $\mathbf{b}$.]

**12W**

A solid shape ABCO is created by an anti-clockwise rotation of part AB of the parabola $z/a = 1 - (x/b)^2$, for which $z \geqslant 0$ and $x \geqslant 0$ (where $a > 0$ and $b \neq 0$), about the $z$-axis by an angle $\alpha$ with $0 < \alpha \leqslant 2\pi$.




(a) What are the cylindrical polar coordinates of an arbitrary point, $(r, \theta, z)$, on the curved surface ABC? [1]

(b) Calculate the volume of the shape ABCO. [3]

(c) Show that the area, $A$, of the curved surface ABC is equal to

$$A = \frac{\alpha b^2}{12}\left(\frac{b^2}{a^2}\right)\left[\left(1 + \frac{4a^2}{b^2}\right)^{3/2} - 1\right].$$

[8]

(d) Find an approximate expression for $A$ in the limiting case $a/b \ll 1$. Comment on your result for $\alpha = 2\pi$. [2]

(e) Calculate the vector area, $\boldsymbol{S}_{\text{ABCA}}$, of the curved part of the surface. [6]

**13T**

(a) Sketch the curve defined in parametric form as

$$x = \frac{3at}{1+t^3}, \quad y = \frac{3at^2}{1+t^3}, \qquad -\infty < t < \infty.$$

Indicate the points on the curve corresponding to $t = 0$, $t = 1$ and $t \to \pm\infty$. Indicate the limiting behaviour of the curve as $t \to -1$. [8]

(b) (i) The curve has a loop which you may treat as closed for the appropriate limits of the parameter. What values of $t$ correspond to the loop? [2]

(ii) Use the formula Area $= \frac{1}{2}\oint (x\mathrm{d}y - y\mathrm{d}x)$ to calculate the area inside the loop. [6]

(c) Obtain the equation of the curve in the form $f(x,y) = 0$, independent of $t$. Find the equation of the line in the $(x,y)$ plane about which the curve is symmetric. [4]

**14W**

(a) Find the general solution of

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{x}{a^2 + x^2}\right) y = x.$$

[6]

(b) (i) Let $v = x + y$ and hence, or otherwise, find the general solution in terms of $x$ and $y$ of the ordinary differential equation,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-(x+y)}{3x + 3y - 4}.$$

[4]

(ii) Do your solutions include a straight line? Justify your answer. [2]

(c) Let $p = \dfrac{\mathrm{d}y}{\mathrm{d}x}$ and consider the ordinary differential equation, where $p \neq \pm 1$,

$$\frac{y}{x} = \frac{2p}{1 - p^2}. \qquad (\dagger)$$

(i) Starting with the differential equation (†) derive the following differential equation relating $p$ and $x$,

$$-p(1 - p^2) = 2x\frac{\mathrm{d}p}{\mathrm{d}x}. \qquad (\dagger\dagger)$$

[2]

(ii) Solve the differential equation (††). [4]

(iii) Substitute your solution to (††) back into (†) to find the curve $y(x)$ which solves

$$\frac{y}{x} = \frac{2p}{1 - p^2}.$$

[2]

**15R**

(a) Consider the system of linear equations, $\mathbf{A}\mathbf{v} = \mathbf{w}$. Describe the possible types of solution by considering first the homogeneous problem $\mathbf{A}\mathbf{v} = 0$ and then the particular solution to the full equation. [6]

(b) Find all the solutions for all real values of the parameter $t$ in the two cases $s = 1$ and $s = 6$ in the matrix equation.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & t & t \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ s \\ 2 \end{pmatrix} .$$

[12]

Interpret the different cases in the example geometrically. [2]

**16X**

(a) An opaque bag labelled $A$ contains 2 green and 3 red balls. Three balls are drawn at random without replacement.

(i) For the number of green balls drawn, list the possible outcomes, calculate their probabilities and show that these sum to one. [3]

(ii) Calculate the expectation value, $\mu$, and the standard deviation, $\sigma$, for the number of green balls. [5]

(iii) What is the probability that the number of green balls lies within one standard deviation, $[\mu - \sigma, \mu + \sigma]$, of the expectation value? [2]

(iv) What are the probabilities of drawing: at least one red ball; at least two red balls? [2]

(b) A second bag labelled $B$ contains 1 green and 2 red balls. Three balls are drawn without replacement from bag $A$ and one ball from bag $B$.

(i) What is the probability that exactly one of these four balls is green? [4]

(ii) Given that exactly one of these four balls is green, what is the probability that it comes from bag $A$? [4]

**17Y**

(a) Verify, by direct substitution, that

$$w(x,y) = \frac{1}{360}(15x^4y^2 - x^6 + 15x^2y^4 - y^6)$$

is a solution of the equation,

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = x^2y^2.$$

[6]

(b) (i) Do the solutions of the homogeneous problem for $u(x,y)$

$$2y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0 \qquad (\dagger)$$

satisfy the principle of superposition? Justify your answer. [2]

(ii) Using the homogeneous equation (†) write down a vector field, $\mathbf{v}(x,y)$, which is perpendicular to the direction of $\nabla u(x,y)$ at the point $(x,y)$. [2]

(iii) Verify that if $\phi(x,y) = x^2 + 2y^2$ then $\nabla\phi$ is a vector field which is proportional to $\nabla u(x,y)$. [2]

(iv) Verify that any function of the form $u(\phi(x,y))$ is a solution of the homogeneous equation (†). [2]

(v) By considering $u_p(x,y) = Ax^my^n$, or otherwise, find a particular solution for the inhomogeneous differential equation

$$2y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = xy(2y^2 - x^2) \qquad (\dagger\dagger).$$

Hence write down the general solution, $u(x,y)$, of the equation (††). [3]

(vi) Hence write down the solution to the inhomogeneous differential equation, (††) satisfying the boundary condition $u(x,1) = x^2$. [3]

**18Z**

(a) A function $f(x)$ is defined on $\left[-\dfrac{L}{2}, \dfrac{L}{2}\right]$. Given that the Fourier series exists, what property is required of $f(x)$ so that the Fourier series may be written as a Fourier sine series in the form,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n x}{L}\right),$$

where

$$b_n = \frac{2}{L}\int_{-L/2}^{+L/2} f(x)\sin\left(\frac{n2\pi x}{L}\right)dx?$$

[2]

(b) Suppose that $f(x)$ can be represented by a Fourier sine series of the form above. Let $g(x)$ be defined as $f(x)$ translated by a strictly positive distance $\ell$ in the positive $x$ direction to give $g(x) = f(x - \ell)$.

(i) Using the Fourier series for $f(x)$ find the form of the Fourier series for $g(x)$. [3]

(ii) Comment on the special case where $\ell$ is an integer multiple of $L$. [2]

(iii) What are the Fourier coefficients of $g(x)$ in terms of the Fourier coefficients of $f(x)$? [3]

(c) (i) Calculate the coefficients of $f(x)$ when $f(x) = Kx/L$ is defined over $-L/2 < x \leqslant L/2$. [4]

(ii) Initially, at a time $t = 0$, the displacement of the function $f(x)$ is given by $\ell = 0$. The function $f(x)$ is translated in the $x$ direction with constant speed $v$. What is the displacement, $\ell(t)$, at a time $t$? Write down the full Fourier series of the moving function in terms of space $x$ and time $t$. [4]

(iii) Can the Fourier series for the space or time derivative of the function be obtained by term-wise differentiation? Justify your answer. [2]

**19Y***

(a) A particle is contained inside a box which has orthogonal sides of length $a$, $b$ and $c$. The particle's energy $E$ is,

$$E = A\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right),$$

where $A$ is a positive constant. By using the method of Lagrange multipliers, or otherwise, determine the shape of the box which minimises the energy $E$, subject to the constraint that the volume is constant. [6]

(b) A thermal nuclear reactor is a circular cylinder, of base radius $R$ and height $H$. For operational reasons the reactor must satisfy the neutron diffusion constraint

$$\phi(R, H) = \left(\frac{2.4048}{R}\right)^2 + \left(\frac{\pi}{H}\right)^2 = const.$$

Demonstrate that the minimal volume is achieved for $\dfrac{H}{R} = \dfrac{\sqrt{2}\pi}{2.4048}$. [5]

(c) Calculate for real and constant $a$,

(i) $\displaystyle\lim_{x\to 0} \frac{\tan x - \sin x}{\sin^3 x}$, [2]

(ii) $\displaystyle\lim_{x\to a} \frac{\sin x - \sin a}{x - a}$, [2]

(iii) $\displaystyle\lim_{x\to\infty} \left(\frac{x+a}{x-a}\right)^x$. [5]

**20T***

(a) Do the solutions of the partial differential equation

$$\frac{\partial^2 f(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 f(x,t)}{\partial t^2} = 0 \qquad (\dagger)$$

satisfy the principle of superposition? Justify your answer. [3]

(b) Using separation of variables, and writing your separation constant in the forms $-\lambda^2$, 0, and $+\lambda^2$, find the general solution of the partial differential equation (†). [7]

(c) Given that the partial differential equation, (†), describes the transverse displacement of a horizontal string with fixed ends at $x = 0$ and $x = L$, write down all the solutions which could describe this situation. Hence write down the general solution of the problem. [5]

(d) Given that the initial conditions are $\dfrac{\partial f(x,0)}{\partial t} = 0$ and $f(x,0) = \sin\dfrac{\pi x}{L}$, write down the expression describing the motion of the string in (c). [5]

## END OF PAPER