UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Monday, 6 June, 2016 9:00 am to 12:00 pm

MATHEMATICS (1)

Before you begin read these instructions carefully:

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

*You may submit answers to **all** of section A, and to no more than **five** questions from section B.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

***Write on one side of the paper only and begin each answer on a separate sheet.** (For this purpose, your section A attempts should be considered as one single answer.)*

Questions marked with an asterisk (*) require a knowledge of B course material.

After the end of the examination:

*Tie up **all of your section A answer** in a single bundle, with a completed blue cover sheet.*

*Each section B question has a number and a letter (for example, **11R**). Answers to each question must be tied up in **separate** bundles and marked (for example **11R, 12Y** etc) according to the number and letter affixed to each question. **Do not join the bundles together**. For each bundle, a blue cover sheet **must** be completed and attached to the bundle, with the correct number and letter written in the box.*

*A **separate** green master cover sheet listing all the questions attempted **must** also be completed. (Your section A answer may be recorded just as A: there is no need to list each individual short question.)*

Every cover sheet must bear your examination number and desk number.
Calculators are not permitted in this examination.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Green master cover sheet
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1

(a) Factorise the expression $a^3 + b^3$. [1]

(b) Find the repeated root of the equation $x^3 + x^2 - 16x + 20 = 0$. [1]

2

(a) Complete the square $-x^2/2 + x$. [1]

(b) Express $\dfrac{x+2}{x^2-1}$ as a sum of real partial fractions. [1]

3

Evaluate,

(a) 10002×9998, [1]

(b) $\ln\left[\dfrac{2}{\pi}\left(\arctan(\pi/4) + \arctan(4/\pi)\right)\right]$. [1]

4

Find all real x obeying the inequality, $\sin(x) \leqslant -\sqrt{3}/2$. [2]

5

(a) Sketch $|y| = 1 - x^2$ in the x-y plane. [1]

(b) Find all the real solutions of the simultaneous equations, $y = 1/2$ and $|y| = 1 - x^2$. [1]

6

Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2$$

satisfying the boundary condition $y = 1$ when $x = 1$. [2]

7

Evaluate each of the following integrals,

(a)

$$\int \frac{1}{\sin^2(3x+1)} \mathrm{d}x \ ,$$

[1]

(b)

$$\int_0^1 (x-2)^n \mathrm{d}x + \int_1^2 (x-2)^n \mathrm{d}x + \int_2^3 (x-2)^n \mathrm{d}x + \ldots + \int_{n-1}^{n} (x-2)^n \mathrm{d}x$$

where n is a non-negative integer. [1]

8

Differentiate $(2e)^{\cos(x)}$ with respect to x. [2]

9

Sketch the graph of the function $y = x^3/(x^3+1)$. Label any asymptotes and find any stationary points. [2]

10

Find graphically, or otherwise, all the values of the real parameter k for which the following set of simultaneous equations has real solutions for (x, y),

$$\begin{aligned} (x-2)^2 + (y-1)^2 &= 1, \\ y &= kx. \end{aligned}$$

[2]

SECTION B

11R

Consider the relation,

$$\left|\frac{z-i}{z^*-i}\right| = \lambda,$$

where $z = x + iy$ is a complex number, z^* is its complex conjugate and λ is a constant.

(a) For $\lambda = 1$, show that the locus is a line in the complex plane and find its equation in terms of x and y. [3]

(b) What is the locus when $\lambda = 0$? [2]

(c) Show that the locus for all other real positive λ may be written in the form $zz^* + bz^* + b^*z + c = 0$ with c a real number. By finding b and c show that this locus is a circle, giving expressions for the centre and radius. [10]

(d) Find conditions, if they exist, on complex b and real c for which the expression $zz^* + bz^* + b^*z + c = 0$, in the compex plane, describes

(i) a point,

(ii) a circle,

(iii) a line. [5]

12Y

(a) The function $f(x, y)$ satisfies the equation

$$y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = 0.$$

Show by changing to new variables, $u = x^2 - y^2$ and $v = 2xy$, that f is a function of $x^2 - y^2$ only. [6]

(b) Let $g(x, y) = 4x^2 + 4y^2 + x^4 - 6x^2y^2 + y^4$.

(i) Locate all the stationary points of the function $g(x, y)$. Classify them as maxima, minima or saddle points. [7]

(ii) Sketch the contours of $g(x, y)$ for x and y such that $0 \leqslant x \leqslant 2$ and $0 \leqslant y \leqslant 2$. You are not required to find equations for the contour lines. [7]

13T

Let C be the curve whose points have position vectors which are given in Cartesian coordinates by

$$\mathbf{r} = (b\cos\theta, b\sin\theta, c\theta),$$

where b and c are positive constants and the parameter θ takes values $0 \leqslant \theta \leqslant 5\pi/2$. Consider the two vector fields, $\mathbf{v_1}$ and $\mathbf{v_2}$, where $\mathbf{a}$ is a non-zero constant vector,

$$\mathbf{v}_1 = \mathbf{a} \times \mathbf{r},$$
$$\text{and} \quad \mathbf{v}_2 = (2xy, x^2 + 3y^2 + z^2, 2yz).$$

(a) Determine $\boldsymbol{\nabla} \cdot \mathbf{v}_1$ and $\boldsymbol{\nabla} \cdot \mathbf{v}_2$. [2]

(b) Evaluate the line integrals

$$I_n = \int_{\mathrm{C}} \mathbf{v}_n \cdot d\mathbf{r}$$

for $n = 1, 2$. [10]

(c) Show that one of the two vector fields $\mathbf{v}_n$ is conservative and that the other is not. [4]

(d) Find a potential ϕ such that the conservative field $\mathbf{v}_n = \boldsymbol{\nabla}\phi$, and verify that the corresponding line integral agrees with the result expected from the values of ϕ at the endpoints. [4]

[Note: $\mathbf{a}\times\mathbf{b}$ and $\mathbf{a}\wedge\mathbf{b}$ are alternative notations for the cross product, or vector product, of two vectors.]

14Z

(a) Let λ_n be the n-th eigenvalue and $\mathbf{e}_n$ the corresponding normalised eigenvector of the real symmetric $N \times N$ matrix $\mathbf{A}$.

(i) Demonstrate that $\mathbf{A}$ can be expressed in the form

$$\mathbf{A} = \sum_{n=1}^{n=N} \lambda_n \mathbf{e}_n \mathbf{e}_n^T = \lambda_1 \mathbf{e}_1 \mathbf{e}_1^T + \lambda_2 \mathbf{e}_2 \mathbf{e}_2^T + ... + \lambda_N \mathbf{e}_N \mathbf{e}_N^T \ . \qquad (\dagger)$$

[2]

(ii) Using equation ($\dagger$) show that the trace of $\mathbf{A}$ is equal to the sum of its eigenvalues. [2]

(iii) Assuming the inverse of $\mathbf{A}$ exists, demonstrate that it takes the form

$$\mathbf{A}^{-1} = \sum_{n=1}^{n=N} \frac{1}{\lambda_n} \mathbf{e}_n \mathbf{e}_n^T.$$

[3]

(iv) Under what circumstances does the inverse not exist? [1]

(b) Suppose that

$$\mathbf{A} = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \qquad (\dagger\dagger)$$

(i) Calculate the eigenvalues and corresponding normalised eigenvectors of $\mathbf{A}$. [6]

(ii) Calculate the inverse matrix, $\mathbf{A}^{-1}$, of the matrix $\mathbf{A}$, ($\dagger\dagger$) above. [4]

(iii) Calculate the trace of the inverse matrix, $\mathbf{A}^{-1}$. Is this the answer which you would expect given the eigenvalues of $\mathbf{A}$? [2]

15T

(a) Find, by any method, the first three non-zero terms in the Taylor series expansion about $x = 0$ of the following functions,

(i) $\dfrac{\ln(2+x)}{2-x}$, [4]

(ii) $\arctan(x)$, [4]

(iii) $\ln(\cosh(x))$. [4]

(b) Consider the function $f(g)$, with $f(0) = 0$ and $\dfrac{\mathrm{d}f}{\mathrm{d}g} \neq 0$.

(i) Write $\dfrac{\mathrm{d}g}{\mathrm{d}f}$ in terms of $\dfrac{\mathrm{d}f}{\mathrm{d}g}$, and then obtain $\dfrac{\mathrm{d}^2 g}{\mathrm{d}f^2}$ and $\dfrac{\mathrm{d}^3 g}{\mathrm{d}f^3}$ in terms of $\dfrac{\mathrm{d}f}{\mathrm{d}g}$, $\dfrac{\mathrm{d}^2 f}{\mathrm{d}g^2}$ and $\dfrac{\mathrm{d}^3 f}{\mathrm{d}g^3}$. [4]

(ii) Given that f can be expressed as a Taylor series of the form

$$f = a_1 g + a_2 g^2 + a_3 g^3 + \cdots\,, \quad |g| \ll 1,$$

find the coefficients b_1, b_2 and b_3 in the inverse series expansion

$$g = b_1 f + b_2 f^2 + b_3 f^3 + \cdots\,, \quad |f| \ll 1.$$

[4]

16X

(a) A traffic control signal has a periodic cycle of period R. Within each cycle the signal is set to stop traffic at times t in the range $0 \leqslant t \leqslant r$ with $0 < r \leqslant R$. At other times in the period the traffic passes without delay. A vehicle is equally likely to approach at any time t within the period $0 \leqslant t \leqslant R$ but not at any other time. Denote the delay to the vehicle by s.

(i) Calculate the probability that the vehicle passes without delay, i.e. $s = 0$. [2]

(ii) For delay strictly greater than zero, i.e. $s > 0$, let $p(s)ds$ represent the probability that the delay is in the range s to $s + ds$. Show that $p(s)$ takes the form

$$p(s) = \begin{cases} \dfrac{1}{R}, & 0 < s \leqslant r, \\ 0, & r < s \leqslant R. \end{cases}$$

[2]

(iii) Sketch the function $p(s)$ for $0 < s \leqslant R$. [2]

(iv) Demonstrate that the probabilities for the vehicle to pass without delay, $s = 0$, or to pass with a finite delay, $0 < s \leqslant r$, sum to unity. [2]

(v) Calculate μ, the mean value for the delay, including contributions from $s = 0$ and $s > 0$. [2]

(vi) Calculate σ, the standard deviation for the delay. [4]

(b) Vehicles A and B approach the signal travelling at the same speed with A leading B by time separation τ_0. Vehicle A arrives at a given time t and has to stop, that is $0 \leqslant t < r$. Vehicle B arrives at time $t + \tau_0$ where τ_0 is a random uniformly distributed variable such that $0 < \tau_0 \leqslant R$. Therefore, in contrast to the situation in (a), B may arrive after the light has changed back to red. After both vehicles have passed, let A lead B by τ_1. Calculate the probabilities for the following outcomes and show that they sum to unity:

(i) $\tau_1 = 0$, [2]

(ii) $0 < \tau_1 < \tau_0$, [2]

(iii) $\tau_1 \geqslant \tau_0$. [2]

[Hint: A diagram showing the arrival of A at a time t and the possible arrival times of B might be of help.]

17Z

Let $y^{(n)} = \dfrac{\mathrm{d}^n y}{\mathrm{d}x^n}$, where n is a positive integer.

(a) Find a general solution of

$$y^{(n)} = 0$$

for

(i) $n = 2$, [1]

(ii) an arbitrary n. [2]

(b) Consider the following equation,

$$y^{(5)} - y^{(1)} = x \ , \qquad (\dagger)$$

which can be solved by trying a solution of the form $y = e^{kx}$ and using the same method as would be used for a second-order ordinary differential equation.

(i) Find the complementary function for equation (†). [6]

(ii) Find a particular integral for equation (†). [3]

(iii) Find the general solution for equation (†). [1]

(iv) Find the solution subject to the initial conditions $y = 1$, $y^{(1)} = 0$, $y^{(2)} = 0$, $y^{(3)} = 0$ and $y^{(4)} = 0$ at $x = 0$ for equation (†). [7]

18S

(a) (i) Define the definite integral $\int_a^b f(x)\,dx$ in terms of a summation over the function evaluated at points x_i in the interval $[a,\ b]$ multiplied by the subintervals $\Delta x_i = x_{i+1} - x_i$ between them. [3]

(ii) Use this definition to approximate the integral

$$I = \int_1^\infty x^{-1-\alpha}\,dx \qquad (\alpha > 0)\,,$$

by evaluating the function at the points $x_i = 1,\ 1+\epsilon,\ (1+\epsilon)^2,\ (1+\epsilon)^3, \ldots$ $(\epsilon > 0)$ for which the subintervals are growing as $\Delta x_i = \epsilon$, $\epsilon(1+\epsilon)$, $\epsilon(1+\epsilon)^2$, $\epsilon(1+\epsilon)^3, \ldots$. Show that this summation is a geometrical series and find the initial term and the common ratio. [4]

(iii) Hence evaluate the integral I.

[Hint: You may find the binomial approximation for $(1+\epsilon)^{-\alpha}$ useful for $\epsilon \ll 1$.] [3]

(b) Let m and n be positive integers. Define the integral,

$$I_{m,n} = \int_0^\infty \frac{x^{m-2}\,dx}{(1+x)^{m+n-2}} \qquad (m > 1,\ n > 1)\,.$$

(i) Use integration by parts, or otherwise, to derive the recurrence formulae,

$$I_{2,n} = \frac{1}{n-1}, \qquad\qquad I_{m,n} = \frac{(m-2)}{(m+n-3)} I_{m-1,n}\ .$$

[5]

(ii) Hence evaluate the integral, $I_{m,n}$. [5]

19Y*

Consider the coordinate transformation from the coordinate system $\{W, \theta, \phi\}$ to the Cartesian coordinate system $\{x, y, z\}$ given by,

$$x = W \tan\theta \cos\phi, \qquad y = W \tan\theta \sin\phi, \qquad z = W,$$

where $0 < x < \infty$, $0 < y < \infty$, $0 < z < \infty$, $0 < W < \infty$, $0 < \theta < \dfrac{\pi}{2}$ and $0 < \phi < \dfrac{\pi}{2}$.

(a) Calculate the partial derivatives,

$$\left(\frac{\partial}{\partial W}\right)_{\theta,\phi}, \qquad \left(\frac{\partial}{\partial \theta}\right)_{W,\phi}, \qquad \left(\frac{\partial}{\partial \phi}\right)_{\theta,W}$$

in terms of the partial derivatives,

$$\left(\frac{\partial}{\partial x}\right)_{y,z}, \qquad \left(\frac{\partial}{\partial y}\right)_{x,z}, \qquad \left(\frac{\partial}{\partial z}\right)_{x,y}.$$

[8]

(b) Write your solution in matrix form as

$$\begin{pmatrix} \left(\frac{\partial}{\partial W}\right)_{\theta,\phi} \\ \left(\frac{\partial}{\partial \theta}\right)_{W,\phi} \\ \left(\frac{\partial}{\partial \phi}\right)_{\theta,W} \end{pmatrix} = \begin{pmatrix} \left(\frac{\partial x}{\partial W}\right)_{\theta,\phi} & \left(\frac{\partial y}{\partial W}\right)_{\theta,\phi} & \left(\frac{\partial z}{\partial W}\right)_{\theta,\phi} \\ \left(\frac{\partial x}{\partial \theta}\right)_{W,\phi} & \left(\frac{\partial y}{\partial \theta}\right)_{W,\phi} & \left(\frac{\partial z}{\partial \theta}\right)_{W,\phi} \\ \left(\frac{\partial x}{\partial \phi}\right)_{\theta,W} & \left(\frac{\partial y}{\partial \phi}\right)_{\theta,W} & \left(\frac{\partial z}{\partial \phi}\right)_{\theta,W} \end{pmatrix} \begin{pmatrix} \left(\frac{\partial}{\partial x}\right)_{y,z} \\ \left(\frac{\partial}{\partial y}\right)_{x,z} \\ \left(\frac{\partial}{\partial z}\right)_{x,y} \end{pmatrix}.$$

[4]

(c) Hence or otherwise find the partial derivatives,

$$\left(\frac{\partial}{\partial x}\right)_{y,z}, \qquad \left(\frac{\partial}{\partial y}\right)_{x,z}, \qquad \left(\frac{\partial}{\partial z}\right)_{x,y}$$

in terms of the partial derivatives,

$$\left(\frac{\partial}{\partial W}\right)_{\theta,\phi}, \qquad \left(\frac{\partial}{\partial \theta}\right)_{W,\phi}, \qquad \left(\frac{\partial}{\partial \phi}\right)_{\theta,W}.$$

[8]

20R*

(a) (i) State a necessary and sufficient condition for a vector field $\mathbf{H}$ to be conservative. [1]

(ii) Given that the vector field $\mathbf{F} = \nabla \times \mathbf{G}$ defines the vector field $\mathbf{G}$, is there more freedom in the definition of $\mathbf{G}$ than just that of an additive constant? Justify your answer. [2]

(iii) Find $\mathbf{F}$ given that $\mathbf{G} = \frac{1}{3}\left(z^3\mathbf{i} + x^3\mathbf{j} + y^3\mathbf{k}\right)$. [2]

(b) (i) With the aid of a diagram state Stokes' theorem. [2]

(ii) Calculate the surface integral

$$A = \int\int_S (y^2\mathbf{i} + z^2\mathbf{j} + x^2\mathbf{k}) \cdot \mathrm{d}\mathbf{S},$$

for the half surface of the ellipsoid, S, given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

with $z \geqslant 0$ and $abc \neq 0$ and with the surface area element pointing outwards from the ellipsoidal surface. [10]

(c) What is the surface integral, A, if the surface S is replaced by the surface,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^8}{c^8} = 1,$$

with $z \geqslant 0$ and $abc \neq 0$ and with the same orientation as the half ellipsoid in part (b)(ii)? [3]

[Note: $\mathbf{a}\times\mathbf{b}$ and $\mathbf{a}\wedge\mathbf{b}$ are alternative notations for the cross product, or vector product, of two vectors $\mathbf{a}$ and $\mathbf{b}$.]

END OF PAPER