UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB & II (General)

Friday, 29 May, 2015 9:00 am to 12:00 pm

MATHEMATICS (2)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, ***6C****).*

Answers must be tied up in ***separate*** *bundles, marked* ***A, B or C*** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A ***separate*** *green master cover sheet listing all the questions attempted* ***must*** *also be completed.*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Green master cover sheet
Script paper

SPECIAL REQUIREMENTS
Calculator - students are permitted to bring an approved calculator.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1A

(a) Explain what it means for the differential operator $\mathcal{L}$ to be self-adjoint on the interval $a \leqslant x \leqslant b$. [2]

The eigenfunctions $y_n(x)$ of a self-adjoint operator $\mathcal{L}$ satisfy

$$\mathcal{L}y_n = \lambda_n w y_n \,,$$

for some weight function $w(x) > 0$. Show that for appropriate boundary conditions, eigenfunctions with distinct eigenvalues are orthogonal, i.e.,

$$\int_a^b w(x) y_m^*(x) y_n(x)\, dx = 0$$

for $\lambda_m \neq \lambda_n$. [4]

(b) Consider the eigenvalue problem

$$-\left(1 - x^2\right) \frac{d^2 y_n}{dx^2} + x \frac{dy_n}{dx} = n^2 y_n \qquad (\star)$$

on the interval $-1 \leqslant x \leqslant 1$, with the boundary conditions $y_n(-1) = 0$ and $y_n(1) = 0$.

(i) Express $(\star)$ in Sturm–Liouville form, and hence determine the weight function $w(x)$. [5]

(ii) By using the substitution $x = \cos\theta$, solve $(\star)$ with the given boundary conditions to show that n must be an integer, and construct the normalised eigenfunctions for $n > 0$. [6]

(iii) Verify explicitly the orthogonality of your eigenfunctions for $n \neq m$. [3]

2B

In plane-polar coordinates (r,θ), Laplace's equation is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}=0\,. \qquad (\star)$$

(a) Use separation of variables to show that the general solution of $(\star)$ that is continuous and single-valued for $r>0$ can be written as

$$\Phi(r,\theta)=A_0+B_0\ln r+\sum_{n=1}^{\infty}\left[\left(A_n r^n+B_n r^{-n}\right)\cos n\theta+\left(C_n r^n+D_n r^{-n}\right)\sin n\theta\right]\,,$$

where A_n, B_n, C_n, and D_n are constants. [10]

(b) The surface of an infinite cylinder is given by $r=R$ in cylindrical polar coordinates (r,θ,z). The cylinder has a surface charge density $\sigma(\theta)$ so the electrostatic potential Φ is continuous at $r=R$, but its normal derivative has a discontinuity:

$$\left(\frac{\partial\Phi}{\partial r}\right)_{r=R^+}-\left(\frac{\partial\Phi}{\partial r}\right)_{r=R^-}=-\sigma(\theta)\,,$$

where R^+ denotes the limit as $r\to R$ from above and R^- the limit as $r\to R$ from below. The surface charge density has Fourier series

$$\sigma(\theta)=\sum_{n=1}^{\infty}\left(a_n\cos n\theta+b_n\sin n\theta\right)\,.$$

Assume that Φ is independent of z and therefore satisfies $(\star)$ for $r<R$ and $r>R$. Determine Φ for all r, assuming that $\Phi\to 0$ as $r\to\infty$ and that Φ is finite at $r=0$. [10]

3B

Let V be a region of three-dimensional space with boundary S.

(a) Prove that

$$\int_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) dV = \int_S \left(\phi\mathbf{n}\cdot\boldsymbol{\nabla}\psi - \psi\mathbf{n}\cdot\boldsymbol{\nabla}\phi\right) dS\,,$$

where ϕ and ψ are scalar fields and $\mathbf{n}$ is the outward-directed unit normal to S. [3]

(b) Let ϕ satisfy Laplace's equation $\nabla^2\phi = 0$ in V, and let $G(\mathbf{x},\mathbf{x}')$ obey

$$-\nabla^2_{\mathbf{x}} G = \delta^{(3)}(\mathbf{x}-\mathbf{x}')\,,$$

where $\boldsymbol{\nabla}_{\mathbf{x}}$ is the gradient with respect to $\mathbf{x}$. Prove that

$$\phi(\mathbf{x}') = \int_S \left[G(\mathbf{x},\mathbf{x}')\mathbf{n}\cdot\boldsymbol{\nabla}_{\mathbf{x}}\phi(\mathbf{x}) - \phi(\mathbf{x})\mathbf{n}\cdot\boldsymbol{\nabla}_{\mathbf{x}}G(\mathbf{x},\mathbf{x}')\right] dS\,.$$

[2]

(c) State the boundary condition that should be imposed on $G(\mathbf{x},\mathbf{x}')$ for it to be a Green's function for Laplace's equation with Dirichlet boundary conditions (i.e., $\phi(\mathbf{x}) = f(\mathbf{x})$ on S). [2]

(d) Let V be the half-space $z > 0$ and let ϕ satisfy Laplace's equation in V with boundary conditions $\phi(x,y,0) = f(x,y)$ and $\phi(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$. Use the method of images to determine $G(\mathbf{x},\mathbf{x}')$ and hence show that, for $z > 0$,

$$\phi(x,y,z) = \frac{z}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(\xi,\eta)}{[(x-\xi)^2+(y-\eta)^2+z^2]^{3/2}}\,d\xi d\eta\,.$$

[9]

[*You may assume that* $H(\mathbf{x}) = \frac{1}{4\pi|\mathbf{x}|}$ *satisfies* $-\nabla^2 H = \delta^{(3)}(\mathbf{x})$.]

(e) Determine $\phi(0,0,z)$ explicitly for the case

$$f(x,y) = \begin{cases} 0 & \text{if } x^2+y^2 > a^2 \\ 1 & \text{if } x^2+y^2 \leqslant a^2, \end{cases}$$

where $a > 0$. [4]

4B

(a) (i) State the residue theorem of complex analysis. [2]

(ii) Consider the function

$$f(z) = \frac{z^2}{1+z^4}\,.$$

State the location of any singularities of $f(z)$ and calculate the residues of $f(z)$ at these singularities, simplifying your answers as much as possible. [7]

(iii) By considering the integral of $f(z)$ around a large semicircle, evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx\,.$$

[3]

(b) Use contour integration to determine the value of

$$\int_{0}^{\infty} \frac{\ln x}{x^2+a^2}\,dx\,,$$

where a is real and positive. State clearly the location of any branch cut required. [8]

5C

The response $y(t)$ of a system to a forcing function $f(t)$ is described by the second-order linear equation

$$\ddot{y} + 2\dot{y} + 5y = f(t)\,. \qquad (\star)$$

You may assume that $f(t)$ vanishes as $t \to \pm\infty$.

(a) By multiplying $(\star)$ by $e^{-i\omega t}$ and integrating, or otherwise, show that the solution to $(\star)$ can be written as

$$\tilde{y}(\omega) = \frac{-\tilde{f}(\omega)}{\omega^2 - 2i\omega - 5}\,,$$

where $\tilde{y}(\omega)$ and $\tilde{f}(\omega)$ are the Fourier transforms of $y(t)$ and $f(t)$, respectively. [5]

(b) Consider the forcing function described by $\tilde{f}(\omega) = i/(\omega - 2i)$.

(i) Use contour integration in the complex ω plane to determine the solution $y(t)$ for both positive and negative t. [12]

(ii) What does this solution imply about $f(t)$ for $t < 0$? (You need not determine $f(t)$ itself.) [3]

6C

Let T be a second-order tensor with components T_{ij} with respect to a Cartesian coordinate system (x_1, x_2, x_3). An alternative Cartesian coordinate system (x'_1, x'_2, x'_3) is defined by $x'_i = M_{ij}x_j$.

(a) What restriction is placed on the transformation matrix M_{ij}? How can one determine whether (x'_1, x'_2, x'_3) is a left- or right-handed coordinate system? Write down expressions for the components of T in the x'_i coordinate system in terms of T_{ij}. [3]

(b) Show that the symmetric and antisymmetric parts of T are second-order tensors. [3]

(c) Consider the second-order tensor field F, with position-dependent components

$$F_{ij} = \begin{pmatrix} x_1^2 & -x_1^2 + x_1x_2 - x_2^2 & x_1 - x_2 \\ x_1^2 + x_1x_2 + x_2^2 & x_2^2 & -x_1 - x_2 \\ -x_1 + x_2 & x_1 + x_2 & 3(x_1^2 + x_2^2) \end{pmatrix}$$

with respect to the x_i coordinates. Write down the components of the symmetric part of F. Determine the principal axes and corresponding principal values of the symmetric part of F, and describe the orientation of the principal axes geometrically. Write down the transformation matrix M_{ij} that is needed to transform from the original axes to these principal axes. [10]

(d) Decompose the tensor field F introduced above as $F_{ij} = P\delta_{ij} + \hat{S}_{ij} + \hat{A}_{ij}$, where P is a scalar field, $\hat{S}_{ij}$ is symmetric and trace-free, and $\hat{A}_{ij}$ is antisymmetric. Determine whether the principal axes of $\hat{S}_{ij}$ are the same as those found in (c). [4]

7C

Three climbers have fallen from an overhanging cliff and are now suspended by their identical elastic safety ropes. The tension in each rope is given by $T(L) = k(L - L_0)$, for $L > L_0$, where k is a constant and L_0 is the unstretched length of each rope. Climber 1 is suspended from the cliff top by rope 1 with stretched length $L_1(t)$. The other two climbers are suspended directly from climber 1. Climber 2 is suspended from climber 1 by rope 2 with stretched length $L_2(t)$, while climber 3 is suspended from climber 1 by rope 3 with stretched length $L_3(t)$. The climbers have masses m_1, m_2, and m_3, respectively. The mass of the ropes is negligible.

(a) Write down expressions for the potential and kinetic energies of the system and hence determine its Lagrangian. (Take the gravitational acceleration to be g and remember to include the elastic potential energy.) [4]

(b) Use the Euler–Lagrange equations to derive the equations of motion for L_i. Show that, at equilibrium, the lengths of the ropes are given by $L_i = \hat{L}_i$ where

$$\begin{aligned} \hat{L}_1 &= L_0 + \tfrac{g}{k}(m_1 + m_2 + m_3)\,, \\ \hat{L}_2 &= L_0 + \tfrac{g}{k}m_2\,, \\ \hat{L}_3 &= L_0 + \tfrac{g}{k}m_3\,. \end{aligned}$$

[5]

(c) Let $y_i = L_i - \hat{L}_i$ be a small departure from equilibrium. Show that

$$\begin{pmatrix} m_1 + m_2 + m_3 & m_2 & m_3 \\ m_2 & m_2 & 0 \\ m_3 & 0 & m_3 \end{pmatrix} \begin{pmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \end{pmatrix} + \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

[2]

(d) Assume, now, that all climbers have equal mass m. Show that one normal mode of oscillation has frequency $\omega = (k/m)^{1/2}$ and that climber 1 is stationary in this mode. For this case, describe the motion of the other two climbers. Determine also the frequencies of the other two modes of oscillation. [9]

8A

(a) Let H be a subgroup of a finite group G. Define the *left coset* gH of H for an element $g \in G$. Prove that the left cosets of H partition G. [6]

(b) Show that the set of all real 3×3 matrices with elements

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \qquad (\star)$$

forms a group under matrix multiplication. Show further that the subset of matrices with $x = z = 0$ forms a normal subgroup. [6]

(c) Now suppose that x, y, and z are integers *mod* 4 (e.g., $5 \, mod \, 4 = 1$). Show that the set of matrices of the form in $(\star)$ is a finite group G under matrix multiplication with arithmetic modulo 4, and determine the order of G. [4]

Show that the subset of such matrices given by $x = z$ defines an Abelian subgroup H. Determine the order of H. How many distinct left cosets of H are there in G? [4]

9B

Let G and G' be finite groups.

(a) Let $\Phi : G \to G'$ be a homomorphism. Define the *kernel* K of Φ. Prove that K is a normal subgroup of G. [5]

(b) Define the *conjugacy class* of $g \in G$. Prove that any normal subgroup of G is a union of conjugacy classes. [3]

(c) What is meant by the *cycle structure* of a permutation? List the possible cycle structures for elements of Σ_3 (the permutation group for three objects). [3]

(d) Assume that $\Phi : \Sigma_3 \to G'$ is a homomorphism that is *onto*, i.e., any element of G' can be written as $\Phi(g)$ for some $g \in \Sigma_3$. Determine the possible forms of K (the kernel of Φ) and hence, or otherwise, prove that G' must be isomorphic to one of Σ_3, C_2 (the cyclic group of order 2), or the trivial group $\{I\}$. [9]

[*You may assume that two elements of* Σ_3 *belong to the same conjugacy class if, and only if, they have the same cycle structure.*]

10C

Consider the D_6 dihedral group

$$G = \{I, R, R^2, R^3, R^4, R^5, m_1, m_2, m_3, m_4, m_5, m_6\} ,$$

with structure defined by the group table

I	R	R^2	R^3	R^4	R^5	m_1	m_2	m_3	m_4	m_5	m_6
R	R^2	R^3	R^4	R^5	I	m_2	m_3	m_4	m_5	m_6	m_1
R^2	R^3	R^4	R^5	I	R	m_3	m_4	m_5	m_6	m_1	m_2
R^3	R^4	R^5	I	R	R^2	m_4	m_5	m_6	m_1	m_2	m_3
R^4	R^5	I	R	R^2	R^3	m_5	m_6	m_1	m_2	m_3	m_4
R^5	I	R	R^2	R^3	R^4	m_6	m_1	m_2	m_3	m_4	m_5
m_1	m_6	m_5	m_4	m_3	m_2	I	R^5	R^4	R^3	R^2	R
m_2	m_1	m_6	m_5	m_4	m_3	R	I	R^5	R^4	R^3	R^2
m_3	m_2	m_1	m_6	m_5	m_4	R^2	R	I	R^5	R^4	R^3
m_4	m_3	m_2	m_1	m_6	m_5	R^3	R^2	R	I	R^5	R^4
m_5	m_4	m_3	m_2	m_1	m_6	R^4	R^3	R^2	R	I	R^5
m_6	m_5	m_4	m_3	m_2	m_1	R^5	R^4	R^3	R^2	R	I

(a) What do the generators R and m_1 represent geometrically? Give an expression for each of the group members in terms of the generators $\{R, m_1\}$. [2]

(b) Identify all the subgroups of order 2 and 3. Are any of these subgroups cyclic? [5]

(c) Explain how to construct a faithful representation of G using 2×2 orthogonal matrices. Give matrices corresponding to R, m_1, and m_2 in such a representation. [4]

(d) Write down the regular representation $D(g)$ for $g = m_4$ and hence or otherwise derive an expression for $[D(m_4)]^n$ for any integer n. [5]

[*Reminder: the regular representation is a set of* $|G| \times |G|$ *permutation matrices each with* $|G|$ *non-zero elements.*]

(e) State the characters of the representations used above in (c) and (d). [4]

END OF PAPER