UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Monday, 8 June, 2015 9:00 am to 12:00 pm

MATHEMATICS (1)

Before you begin read these instructions carefully:

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

*You may submit answers to **all** of section A, and to no more than **five** questions from section B.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

***Write on one side of the paper only and begin each answer on a separate sheet.** (For this purpose, your section A attempts should be considered as one single answer.)*

Questions marked with an asterisk (*) require a knowledge of B course material.

After the end of the examination:

*Tie up **all of your section A answer** in a single bundle, with a completed blue cover sheet.*

*Each section B question has a number and a letter (for example, **11X**). Answers to each question must be tied up in **separate** bundles and marked (for example **11X, 12T** etc) according to the number and letter affixed to each question. **Do not join the bundles together**. For each bundle, a blue cover sheet **must** be completed and attached to the bundle, with the correct number and letter written in the box.*

*A **separate** green master cover sheet listing all the questions attempted **must** also be completed. (Your section A answer may be recorded just as A: there is no need to list each individual short question.)*

Every cover sheet must bear your examination number and desk number.
Calculators are not permitted in this examination.

STATIONERY REQUIREMENTS	***SPECIAL REQUIREMENTS***
6 blue cover sheets and treasury tags	*None*
Green master cover sheet	
Script paper	

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1

(a) Factorise the expression $x^3 - x^2 - x + 1$, and [1]

(b) express its reciprocal as a sum of partial fractions. [1]

2

The line $y = mx$, where m is a positive constant, has only a single point of contact with the curve $y = \mathrm{e}^x$. Calculate

(a) the value of m, and [1]

(b) the point of contact. [1]

3

In the (x, y) plane sketch and label the locus defined by

$$y^2 + x^2 - 2x = 3\,.$$

[2]

4

(a) Calculate the second derivative with respect to x of the real function $y = \ln(\sin x)$. [1]

(b) Sketch the function $y = \ln(\sin x)$ in the range $0 < x < \pi$. [1]

5

Calculate the indefinite integral

$$\int x^3 \cos(x^2)\,\mathrm{d}x$$

and the definite integral

$$\int_0^{\sqrt{\pi/2}} x^3 \cos(x^2)\,\mathrm{d}x\,.$$

[2]

6

Given $x = \cos(t)$ and $y = \sin(3t)$, where $0 \leqslant t < 2\pi$,

(a) calculate the maximum value of y and the corresponding value or values of x, and [1]

(b) using the chain rule, calculate $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ as a function of t. [1]

7

(a) Write down the first three non-zero terms of the expansion about $x = 0$ of $y = \sqrt{(1-3x)}$. [1]

(b) For what range of values of x is this expansion valid? [1]

8

Solve the following equation for x in the range $-\pi < x < \pi$, with $x \neq 0$:

$$\cot(x) = \sin(2x)\,.$$

[2]

9

(a) Find the gradient $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at the point where $x > 0$ and $y = 1$ for the curve defined by

$$y^2 + 2x^2 = 4\,.$$

[1]

(b) Find the equation of the normal to the curve at this point. [1]

10

Evaluate

(a)

$$N_1 = 1 + 2 + 3 + \cdots + 999 + 1000\,,$$

[1]

(b)

$$N_2 = 2 + 4 + 8 + \cdots + 1024 + 2048\,.$$

[1]

SECTION B

11R

(a) Let z be a complex number and let $w = \dfrac{z - \mathrm{i}}{z + \mathrm{i}}$.

(i) Evaluate w when $z = 0$, and when $z = 1$.

(ii) Let $z = \lambda$ where λ is real. Show that for any such z the corresponding w always has unit modulus. [4]

(b) Let $z(t) = r\,\mathrm{e}^{\mathrm{i}\theta}$, where both $r(t)$ and $\theta(t)$ are functions of a real parameter t. Denote the derivatives with respect to the parameter t with a dot. Find $\dot{z}(t)$ and $\ddot{z}(t)$ in terms of $r(t)$ and $\theta(t)$ and their derivatives with respect to t. In each case write your answer in the form $(a(t) + \mathrm{i}\,b(t))\,\mathrm{e}^{\mathrm{i}\theta(t)}$. [4]

(c) Assume that a point moves in the Argand plane and has the position $z(t)$ at a time t. The velocity and acceleration are then $\dot{z}(t)$ and $\ddot{z}(t)$, respectively.

(i) By simplifying the equation $|z - 4| = 2|z - 1|$ identify this locus and sketch it in the Argand plane. [2]

(ii) Assume that the point's path $z(t)$ is on the locus $|z - 4| = 2|z - 1|$. For an arbitrary position on this locus, sketch a vector representing the velocity of the point moving in the anticlockwise direction on the locus. What can you deduce about $\dot{r}$ and $\ddot{r}$? Write down expressions for $\dot{z}(t)$ and $\ddot{z}(t)$, as derived in (b), for this particular motion. [5]

(iii) Assume, instead, that the point moves in the Argand plane with an arbitrary path $z(t)$. Write down the radial and transverse components of the velocity and acceleration with respect to the origin. (The transverse velocity is perpendicular to the line joining the origin to the moving point.)

[*Hint: Think of the significance of two vectors in the Argand plane which can be represented by* $\mathrm{e}^{\mathrm{i}\theta}$ *and* $\mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta}$.] [5]

12S

(a) Determine whether the following differential forms are exact. For each one that is exact, find a function f such that the differential form is equal to $\mathrm{d}f$.

(i) $\exp(x+y)\,\mathrm{d}x + \exp(x+y)\,\mathrm{d}y\,,$

(ii) $\sin x \sin y\,\mathrm{d}x + \cos x \cos y\,\mathrm{d}y\,,$

(iii) $2xy^3z^4\,\mathrm{d}x + 3x^2y^2z^4\,\mathrm{d}y + 4x^2y^3z^3\,\mathrm{d}z\,.$

[9]

(b) Find and classify all the stationary points of the function

$$g(x,y) = 1 - \cos x + \tfrac{1}{2}y^2\,,$$

and calculate the stationary values of g.

Sketch the contours of $g(x,y)$ in the region $-2\pi < x < 2\pi$, $-3 < y < 3$, paying particular attention to any contour lines that pass through the stationary points, and labelling the important features of the plot. [11]

13S

The function $u(x,t)$ satisfies the partial differential equation

$$\frac{\partial u}{\partial t} = \lambda \frac{\partial^2 u}{\partial x^2} \qquad (\dagger)$$

in $-\infty < x < \infty$, where λ is a positive constant.

(a) Given that equation (†) has a solution of the form

$$u(x,t) = (t+a)^{-1/2}\, v(y)\,,$$

valid for $t > -a$, where

$$y = (t+a)^{-1/2}(x+b)$$

and a and b are arbitrary constants, show that the function $v(y)$ satisfies the ordinary differential equation

$$-\frac{1}{2}\left(v + y\frac{\mathrm{d}v}{\mathrm{d}y}\right) = \lambda \frac{\mathrm{d}^2 v}{\mathrm{d}y^2}\,.$$

[6]

(b) Verify that this ordinary differential equation has a solution of the form

$$v(y) = \exp(-cy^2)$$

if the constant c is chosen appropriately. Write the corresponding function $u(x,t)$ explicitly. [4]

(c) Which properties of equation (†) allow the principle of superposition to be applied? Find (in terms of x, t and λ) the solution of equation (†) for $t > 0$ given the initial condition

$$u(x,0) = \exp\left[-(x+1)^2\right] + \exp\left[-(x-1)^2\right]\,.$$

For what range of $t < 0$ is the solution also valid? [10]

14T

A curve C is defined in terms of the parameter t as

$$x = a(t - \sin t)\,, \qquad y = a(1 - \cos t)\,,$$

where a is a positive constant and $0 < t < 2\pi$.

(a) Determine $\dfrac{\mathrm{d}x}{\mathrm{d}t}$, $\dfrac{\mathrm{d}y}{\mathrm{d}t}$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ as functions of t. [3]

(b) Sketch the curve in the (x, y) plane. [4]

(c) Find the point (x, y) on the curve at which the tangent to the curve has slope $\sqrt{3}$. [3]

(d) Determine the area between the curve and the x axis. [5]

[*Hint: You may wish to express the area as an integral with respect to* t.]

(e) Determine the length of the curve. [5]

15T

(a) State Taylor's theorem for the expansion about $x = a$ of a function that is differentiable n times, giving the first three terms explicitly, together with an expression for the remainder term R_n after n terms. [4]

(b) Find, by any method, the first three non-zero terms in the expansion about $x = 2$ of the function

$$\cosh(\sqrt{x})\,.$$

[6]

(c) Find, by any method, the first three non-zero terms in the expansions about $x = 0$ of

(i)
$$\frac{\sin x}{(1+x)^2}\,,$$

[4]

(ii)
$$\frac{x \sin x}{\ln(1+x^2)}\,.$$

[6]

[You may quote standard power series without proof.]

16W

(a) A box contains 3 white (W) and 4 black (B) balls. Balls are taken randomly from the box without replacement. Using the notations $P(\mathrm{W}_i)$ for the probability to withdraw a white ball at the i-th withdrawal and $P(\mathrm{B}_j)$ for the probability to withdraw a black ball at the j-th withdrawal, find

(i) $P(\mathrm{W}_1)$, $P(\mathrm{B}_2)$, $P(\mathrm{W}_3)$, [4]

(ii) $P(\mathrm{B}_1 \cap \mathrm{B}_2)$, [1]

(iii) $P(\mathrm{B}_2 \cap \mathrm{B}_3)$, $P(\mathrm{B}_2|\mathrm{B}_3)$, $P(\mathrm{B}_3|\mathrm{B}_2)$, [5]

(iv) $P(\mathrm{B}_1 \cup \mathrm{B}_2)$, [2]

(v) the expectation value of the number of black balls, $\mathbb{E}[N(\mathrm{B}_3)]$, taken from the box after three withdrawals. [3]

(b) A box contains $N > 1$ white and black balls, of which N_B $(0 < N_\mathrm{B} \leqslant N)$ are black. Again, balls are taken randomly from the box without replacement. Find $P(\mathrm{B}_n)$ for $1 \leqslant n \leqslant N$. [5]

17Y

(a) Evaluate from first principles, by considering elementary areas, the integral

$$I = \int_0^b x^2 \, \mathrm{d}x \, .$$

[You may assume that $\displaystyle\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1)$.] [6]

(b) Evaluate

(i)

$$\int \frac{\mathrm{d}x}{\sqrt{4 - x^2}} \, ,$$

[3]

(ii)

$$\int \frac{(x+1)\,\mathrm{d}x}{x^2 + 4x + 8} \, .$$

[5]

(c) Find the recurrence relation between $F(k)$ and $F(k-2)$, valid for $k \geqslant 1$, where

$$F(k) = \int_0^{\pi} x^k \sin x \, \mathrm{d}x \, ,$$

and use it to calculate $F(k)$ for the case $k = 5$. [6]

18Z

Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and the unit vector

$$\boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} .$$

(a) Calculate the effect of applying $\mathbf{A}$ once, twice and three times to $\boldsymbol{x}$. [3]

(b) Calculate the transpose of $\mathbf{A}$, and the effect of applying it once, twice and three times to $\boldsymbol{x}$. [3]

(c) Considering the observed effects of $\mathbf{A}$ and $\mathbf{A}^{\mathrm{T}}$, why does it follow that $\mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{I}$ and $\mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{I}$? [2]

(d) Calculate the eigenvalues of $\mathbf{A}$ in complex polar form, and the eigenvectors. [4]

[The eigenvalues and eigenvectors of a non-symmetric matrix are calculated in the same way as those of a symmetric matrix, but may be complex. You are not required to normalize the eigenvectors.]

(e) Calculate the eigenvalues of $\mathbf{A}^{\mathrm{T}}$ in complex polar form, and the eigenvectors. [4]

(f) How is the symmetric part of the matrix $\mathbf{A}$ defined in terms of $\mathbf{A}$ and $\mathbf{A}^{\mathrm{T}}$? [1]

(g) Use the eigenvalues of $\mathbf{A}$ and $\mathbf{A}^{\mathrm{T}}$ to calculate the eigenvalues of the symmetric part of $\mathbf{A}$ without calculating the symmetric part of $\mathbf{A}$ explicitly. [3]

19Y*

(a) Find the sum of the first N terms of the following series. Deduce that the infinite series converges, and determine its value.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{3^n}$$

[7]

(b) Use de Moivre's theorem to show that

$$\cos\theta + \cos(\theta+\alpha) + \cdots + \cos(\theta+n\alpha) = \frac{\sin\left[\frac{1}{2}(n+1)\alpha\right]}{\sin\left(\frac{1}{2}\alpha\right)} \cos(\theta + \tfrac{1}{2}n\alpha)\,.$$

[8]

(c) Show that the series

$$\sum_{n=2}^{\infty} \ln\left[\frac{n+(-1)^n}{n}\right]$$

is only conditionally convergent. [5]

20Z*

The velocity field of a body of incompressible liquid takes the form

$$\boldsymbol{v} = -\alpha yz\,\boldsymbol{i} + \alpha xz\,\boldsymbol{j}\,,$$

where x, y and z are Cartesian coordinates, $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are the corresponding unit vectors, and α is a positive constant.

(a) Sketch a vector diagram showing the spatial three-dimensional form of the flow, and label any salient features. [3]

[*Hint: Consider the vector field in a plane* $z =$ constant.]

(b) State the divergence theorem, and by calculating the divergence of $\boldsymbol{v}$ show that there is no net flow across any closed surface within the liquid. [4]

(c) Calculate the curl of $\boldsymbol{v}$ at a general point and express the result in terms of the unit vector $\boldsymbol{k}$ and a radial unit vector $\hat{\boldsymbol{\rho}}$ that is perpendicular to $\boldsymbol{k}$ and directed away from the z axis. [4]

(d) Consider a closed cylindrical surface, $\mathcal{S}$, having radius R and centred on the z axis. If the ends of the cylinder are at $z = \pm h/2$, show explicitly that

$$\int_{\mathcal{S}} (\boldsymbol{\nabla} \times \boldsymbol{v}) \cdot \mathrm{d}\boldsymbol{S} = 0\,.$$

[5]

(e) An arbitrarily shaped hole is opened up on the curved surface of the cylinder, such that an area A of the curved surface is removed. State Stokes's theorem and thereby derive an expression for the line integral of the velocity field around the edge of the hole. [4]

END OF PAPER