UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB & II (General)

Friday, 30 May, 2014 9:00 am to 12:00 pm

MATHEMATICS (2)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, ***6C****).*

Answers must be tied up in **separate** *bundles, marked* ***A, B or C*** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A ***separate*** *green master cover sheet listing all the questions attempted* ***must*** *also be completed.*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Green master cover sheet
Script paper

SPECIAL REQUIREMENTS
Calculator - students are permitted to bring an approved calculator.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1A

(i) The inner product of two functions $f(x)$ and $g(x)$, defined on the closed interval $[a, b]$, is

$$\langle f|g\rangle \equiv \int_a^b f^* g\, w\, dx \ ,$$

where $w(x) > 0$. Consider the operator

$$\mathcal{L} \equiv -\frac{1}{w(x)}\left[\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) - q(x)\right] \ , \qquad a \leqslant x \leqslant b \ ,$$

where $p(x) > 0$.

(a) Derive the boundary conditions under which $\mathcal{L}$ is self-adjoint over the range $[a, b]$, with respect to the inner product defined above. [3]

(b) Show that any two eigenfunctions of $\mathcal{L}$ with distinct eigenvalues are orthogonal. [3]

(ii) Consider the eigenvalue problem

$$\mathcal{L}y \equiv -x^2 y'' - x y' - y = \lambda y \ , \qquad (\star)$$

with boundary conditions $y(1) = y(e) = 0$.

(a) Show that $(\star)$ can be written in Sturm–Liouville form and identify the functions $p(x)$, $q(x)$ and $w(x)$. [2]

(b) Find the eigenvalues and orthonormal eigenfunctions of $\mathcal{L}$. [6]

(c) Derive the solution to the inhomogeneous equation $\mathcal{L}y = 1$ as an eigenfunction expansion. [6]

2B

(i) Let $\Psi(r,\theta)$ be an axisymmetric solution of Laplace's equation in spherical polar coordinates,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right)=0\,.$$

By the method of separation of variables, derive the general solution

$$\Psi(r,\theta)=\sum_{\ell=0}^{\infty}\left(a_\ell r^\ell+\frac{b_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)\,.$$

Here, $P_\ell(\cos\theta)$ is the ℓth Legendre polynomial, i.e., the solution of the differential equation

$$\frac{d}{dx}\left((1-x^2)\frac{dP_\ell}{dx}\right)+\ell(\ell+1)P_\ell=0\,,$$

with $x=\cos\theta$, which is regular at $x=\pm1$. [8]

(ii) A surface charge density $\sigma(\theta)=A\sin^2\theta$ lies on the surface of a sphere of radius R centred on the origin. The electrostatic potential $\Psi(r,\theta)$ satisfies Laplace's equation for $r\neq R$, is continuous and regular everywhere, and tends to zero as $r\to\infty$. The surface charge causes a discontinuity in the radial gradient of Ψ across $r=R$ given by

$$\lim_{\epsilon\to0}\left(\left.\frac{\partial\Psi}{\partial r}\right|_{R+\epsilon}-\left.\frac{\partial\Psi}{\partial r}\right|_{R-\epsilon}\right)=-\sigma\,.$$

Determine Ψ for $r<R$ and $r>R$. [12]

[*Note:* $P_0(x)=1$ *and* $P_2(x)=(3x^2-1)/2$.]

3B

(i) Two scalar functions $\phi(\mathbf{r})$ and $\psi(\mathbf{r})$ are defined in a volume V of three-dimensional space with boundary S. Show that

$$\int_V \left[\phi\nabla^2\psi - \psi\nabla^2\phi\right]\, dV = \int_S \left[\phi\,\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\psi - \psi\,\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}\phi\right]\, dS\,,$$

where $\widehat{\mathbf{n}}$ is the outward-directed unit normal to S. [3]

(ii) Suppose that $\phi(\mathbf{r})$ satisfies

$$\nabla^2\phi + k^2\phi = 0\,,$$

for some real and positive k.

(a) Introducing the Green's function $G(\mathbf{r},\mathbf{r}')$ that satisifes

$$\nabla^2 G(\mathbf{r},\mathbf{r}') + k^2 G(\mathbf{r},\mathbf{r}') = \delta^{(3)}(\mathbf{r}-\mathbf{r}')\,,$$

show that

$$\phi(\mathbf{r}') = \int_S \phi(\mathbf{r})\widehat{\mathbf{n}}\cdot\boldsymbol{\nabla}G(\mathbf{r},\mathbf{r}')\, dS$$

for $\mathbf{r}'$ in V and a suitable boundary condition for $\mathbf{r}$ on S that you should specify. For the case that V is all space, show that a suitable Green's function is

$$G(\mathbf{r},\mathbf{r}') = A\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\,,$$

where the constant A should be determined. [10]

(b) Determine the Green's function for the case that V is the half-space $z \geqslant 0$. Assuming that ϕ falls to zero sufficiently rapidly as $|\mathbf{r}| \to \infty$, show that

$$\phi(\mathbf{r}') = -\frac{ik}{2\pi}\int_{z=0} \frac{e^{ikR}}{R}\left(1+\frac{i}{kR}\right)\cos\theta\,\phi(\mathbf{r})\, dS\,,$$

where R is the magnitude of $\mathbf{R} \equiv \mathbf{r}' - \mathbf{r}$, which makes an angle θ with the positive z-direction, and the integral is over the plane $z = 0$. [7]

4B

(i) For real a and b, with $a > b > 0$, show that

$$z^2 + 2i(a/b)z - 1 = 0$$

has a single solution within the unit circle $|z| = 1$ in the complex plane. [4]

By evaluating a suitable contour integral, show that

$$\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

for real a and b, with $a > b > 0$. [6]

(ii) By integrating the complex function

$$f(z) = \frac{\ln(z+i)}{z^2+1}$$

along the real axis, evaluate the real integral

$$\int_0^{\infty} \frac{\ln(x^2+1)}{x^2+1}\, dx\,.$$

[10]

5C

The Fourier transform in x of a function $u(x,t)$ is given by

$$\tilde{u}(k,t) = \int_{-\infty}^{\infty} u(x,t)e^{-ikx}dx\,. \qquad (\star)$$

(i) Consider the following partial differential equation for $u(x,t)$:

$$\frac{\partial^2 u}{\partial t^2} + 2\gamma\frac{\partial u}{\partial t} + \gamma^2 u = c^2\frac{\partial^2 u}{\partial x^2}\,, \qquad (\star\star)$$

where γ and c are real constants. Write down the corresponding ordinary differential equation for $\tilde{u}(k,t)$, defined in $(\star)$. You may assume that u and its derivatives vanish as $|x| \to \infty$. [2]

(ii) Seeking solutions of the form e^{rt} for constant r, find the general solution to the Fourier transform of $(\star\star)$ for $\tilde{u}(k,t)$, and hence find the general solution for $u(x,t)$. [8]

(iii) Solve $(\star\star)$ for $u(x,t)$ subject to the following initial conditions at $t=0$:

$$u = e^{-|x|} \quad \text{and} \quad \frac{\partial u}{\partial t} = 0\,.$$

[10]

6C

(i) Write down the transformation law for a tensor of order n. Use this to define an isotropic tensor. [2]

(ii) Consider a three-dimensional vector field with Cartesian components u_i. Show that $\partial u_i/\partial x_j$ is an order 2 tensor. [4]

(iii) Write down the transformation law for an axial vector. Under what conditions does an axial vector obey the same transformation law as a vector? Show that the curl of u_i is an axial vector field. [6]

(iv) Show that $\partial u_i/\partial x_j$ can be decomposed into the following terms

$$\frac{\partial u_i}{\partial x_j} = p\delta_{ij} + s_{ij} + \epsilon_{ijk}\omega_k \,, \qquad (\star)$$

where s_{ij} is a symmetric, traceless tensor, ω_k is an axial vector field, ϵ_{ijk} is the Levi–Civita symbol, and δ_{ij} is the Kronecker delta. Find p, s_{ij}, and ω_k, expressed in terms of u_i. [4]

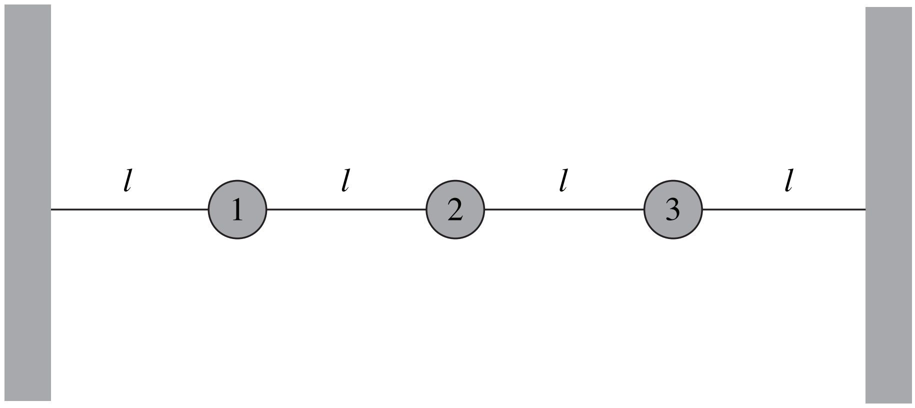
(v) Consider the three-dimensional vector field

$$u_i = (ax_2, bx_1, 0)\,,$$

where a and b are constants. Find ω_k and the principal values and principal axes of s_{ij}, where ω_k and s_{ij} are defined in $(\star)$. [4]

7C

A loaded string, sketched below, consists of a string stretched tightly between two vertical walls with three beads of equal mass m, numbered 1, 2, and 3 as shown, attached at regular intervals with spacing l. Assume the beads are constrained to move vertically and that the tension in the string, τ, is positive and constant. Let z_i be the upward displacement of the ith bead (neglect gravity).



(i) For small displacements, $|z_i| \ll l$, the potential energy, V, stored in the string is

$$V = \frac{\tau}{2l}\left(z_1^2 + (z_1 - z_2)^2 + (z_2 - z_3)^2 + z_3^2\right).$$

Find the normal modes of oscillation and their associated frequencies. Sketch the displacements associated with each normal mode. [12]

(ii) At time $t = 0$, bead 2 is displaced upwards by a distance a, so that $z_2 = a$, while the other beads are at their equilibrium positions ($z_1 = z_3 = 0$), and all beads are initially at rest. Find the subsequent time evolution of the displacement of each bead, and describe the motion in terms of the normal modes. [8]

8A

(i) Let G be a finite group. The *centre* $Z(G)$ of G is the set of elements $z \in G$ that commute with every element $g \in G$, i.e.,

$$Z(G) = \{z \in G \,|\, gz = zg, \forall\, g \in G\} \ .$$

Prove that $Z(G)$ is a subgroup of G. [8]

(ii) Let C_n be the order n cyclic group.

Determine whether the product group $C_2 \times C_3$ is isomorphic to the group C_6. Do the same for $C_2 \times C_4$ and C_8. [8]

What condition do the integers n and m have to satisfy in order for $C_n \times C_m$ to be isomorphic to $C_{n \times m}$? [4]

9B

(i) State Lagrange's theorem relating the order of a group to the orders of its subgroups. [2]

(ii) The symmetry group D_N of a regular N-sided polygon is generated by elements R and m, with $R^N = I$, $m^2 = I$ and $Rm = mR^{-1}$.

(a) List the distinct group elements of D_5 and indicate the geometric action of all order 2 elements on a sketch. [5]

(b) Find all proper subgroups of D_5. [4]

(c) Explain the notion of a conjugacy class of a finite group and determine the conjugacy classes of D_5. Determine which of the proper subgroups of D_5 are normal. [9]

10C

(i) Explain what is meant by a *representation* D of a group G. Define the terms *faithful representation*, *equivalent representation*, and the *character* of a representation. [4]

(ii) Construct the group table for the order 4 cyclic group $C_4 = \{I, a, a^2, a^3\}$. [4]

(iii) Consider the following faithful representations of C_4:

$$D_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad D_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad D_3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad D_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad E_2 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \quad E_3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad E_4 = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}.$$

Determine whether the representations D and E are *equivalent* or *inequivalent*, clearly justifying your answer. Find the characters of each representation. [4]

(iv) Consider a three-dimensional representation, T, of C_4 for which the element a is represented by

$$T(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & b \\ 0 & c & 0 \end{pmatrix}.$$

What are the conditions on the real constants b and c such that T is: (1) a *faithful* representation; and (2) an *unfaithful* representation of C_4? [8]

END OF PAPER