UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Wednesday, 11 June, 2014 9:00 am to 12:00 pm

MATHEMATICS (2)

Before you begin read these instructions carefully:

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

You may submit answers to ***all*** *of section A, and to no more than* ***five*** *questions from section B.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on one side of the paper only and begin each answer on a separate sheet. *(For this purpose, your section A attempts should be considered as one single answer.)*

Questions marked with an asterisk (*) require a knowledge of B course material.

After the end of the examination:

Tie up ***all of your section A answer*** *in a single bundle, with a completed blue cover sheet.*

Each section B question has a number and a letter (for example, ***11S****). Answers to each question must be tied up in* ***separate*** *bundles and marked (for example,* ***11S, 12X*** *etc) according to the number and letter affixed to each question.* ***Do not join the bundles together****. For each bundle, a blue cover sheet* ***must*** *be completed and attached to the bundle, with the correct number and letter written in the box.*

A ***separate*** *green master cover sheet listing all the questions attempted* ***must*** *also be completed. (Your section A answer may be recorded just as A: there is no need to list each individual short question.)*

Every cover sheet must bear your examination number and desk number.
Calculators are not permitted in this examination.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Green master cover sheet
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1

Two vectors are given by $\mathbf{u} = (a,\, 1,\, -3)$ and $\mathbf{v} = (2,\, 2,\, b)$, where a and b are constants. If $\mathbf{u} \times \mathbf{v}$ lies in the x-y plane and $\mathbf{u} \cdot \mathbf{v} = 0$, what values must a and b take? [2]

[*Notation:* $\mathbf{u} \times \mathbf{v}$ *is equivalent to* $\mathbf{u} \wedge \mathbf{v}$.]

2

Give the real and imaginary parts of

$$\cosh(\alpha + \mathrm{i}\beta),$$

where α and β are real numbers. [2]

3

Find the first two non-zero terms in the Taylor expansion of

$$f(x) = \mathrm{e}^{x} \ln(1 + x)$$

around $x = 0$. [2]

4

Write down the complementary function and a particular integral of the ordinary differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 3x\,.$$

[2]

5

Consider the vector field

$$\mathbf{F} = \left(\mathrm{e}^{-x}\cos z\right)\mathbf{i} + \left(\mathrm{e}^{-y}\sin z\right)\mathbf{j} + \mathbf{k}.$$

Calculate both the divergence and the curl of $\mathbf{F}$. [2]

6

Show that $u = f(x - ct)$ is a solution to the partial differential equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0,$$

where c is a constant and f is an unspecified differentiable function of one variable. [1]

If $u = \cos x$ at $t = 0$, what is u at time $t > 0$? [1]

7

What are the two eigenvalues of

$$\begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix},$$

where ϕ is a real constant? [2]

8

A surface is described by the equation $z = \cos x \sin y$, where $0 < x < 2\pi$ and $0 < y < 2\pi$. Find the coordinates (x, y) of any two of its stationary points. [2]

9

Consider the two-dimensional line integral

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r},$$

where $\mathbf{F} = (x^2 + y^2)\mathbf{i}$ and $\mathcal{C}$ is a path in the x-y plane. Calculate the line integral when

(a) $\mathcal{C}$ is the straight line going from $(0, -1)$ to $(0, 1)$, [1]

(b) $\mathcal{C}$ is the straight line going from $(1, 0)$ to $(-1, 0)$. [1]

10

A continuous random variable X takes values 1 and greater. Its normalised probability distribution is

$$f(x) = \alpha\, x^{-\alpha-1},$$

where $\alpha > 1$. Evaluate the following probabilities:

(a) $P(\, X \geqslant 1\,)$, [1]

(b) $P(\, 2 \leqslant X \leqslant 3\,)$. [1]

SECTION B

11S

(a) Addenbrooke's Hospital is conducting a clinical drug trial. The study randomly draws patients from a large population and randomly places them in either a 'control group' or a 'treatment group', each of the same size. The probability that an adverse event occurs to someone in the control group is p_0, and to someone in the treatment group is p_1.

Bruce is taking part in the clinical trial and suffers an adverse event. What is the probability that he is in the control group? [6]

(b) The Rayleigh distribution of a positive continuous random variable X is defined by

$$f(x) = A\,x \exp\left(-\frac{x^2}{2\sigma^2}\right),$$

where σ is a positive parameter and A is a normalisation constant.

Find the value of A in terms of σ. [2]

Sketch the distribution $f(x)$. [2]

What is the mean of the distribution? [2]

(c) Sheila and Bruce are playing a match of table tennis. The first player to win 11 games wins the match. The probability that Sheila wins a game is θ.

What is the probability that Sheila wins m games and loses n games, where both m and n are integers less than 11? [4]

Hence show that the probability that Sheila wins the match is

$$\sum_{n=0}^{10} \binom{10+n}{10} \theta^{11}(1-\theta)^n .$$

[4]

[*Recall the definition* $\binom{N}{n} = N!/(n!(N-n)!)$.]

12X

(a) (i) What is an orthogonal matrix? [2]

(ii) Calculate $\mathbf{M}^{-1}$ where

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -3 & 0 \\ 1 & 1 & 1 \end{pmatrix} .$$

Is $\mathbf{M}$ orthogonal? [6]

(iii) Given that $\mathbf{A}$ and $\mathbf{B}$ are orthogonal matrices, show that $\mathbf{AB}$ is also orthogonal. [2]

(b) (i) Express the following set of simultaneous equations, in which a is a real constant, in matrix form $\mathbf{Ax} = \mathbf{y}$:

$$\begin{aligned} x + y + z &= 3 \\ 2x + 2y + (a-1)z &= 4 \\ ax + y + 3z &= 5 \end{aligned}$$

[2]

(ii) Use matrix methods to solve for the special case $a = 2$. [3]

(iii) Use matrix methods to find values of a for which the equations have no solutions, or multiple solutions. [3]

(iv) Solve for the case which gives rise to multiple solutions. [2]

13W

(a) Evaluate the following indefinite integrals:

(i)

$$\int \left(\cosh^2 x + \cosh x - \sinh^2 x + \sinh x\right)\ dx\,,$$

[3]

(ii)

$$\int \frac{x-1}{3x^2+2x+2}\ dx\,.$$

[8]

(b) Evaluate the definite integral:

$$\int_{-1/\pi}^{1/\pi} \sin^2\left(3x^3+2x\right) \ln\left[\frac{1-x^5}{1+x^5}\right]\ dx\,.$$

[2]

(c) State the fundamental theorem of calculus and thus find

$$\frac{d}{dx}\left[\int_a^x f(y)\ dy\right]\,,$$

where a is a real constant. [2]

(d) Find

$$\frac{d}{dx}\left[\sum_{n=0}^{N}\binom{N}{n}\int_n^x \sin(y^2+y^6)\ dy\right]\,.$$

[5]

[*Recall the definition* $\binom{N}{n} = N!/(n!(N-n)!)$.]

14Y

(a) Show that the differential $(4x + xy^2)dx + (y + x^2y)dy$ is exact and find a function $f(x, y)$ such that

$$df = (4x + xy^2)dx + (y + x^2y)dy\,.$$

Use your result to solve $df = 0$ for $y(x)$. [4]

Find the particular solution for $y(x)$ given that $y(1) = 2$. [2]

(b) It is given that

$$M(x, y)dx + N(x, y)dy = 0$$

has an integrating factor μ which depends only on x. By writing μ in the form

$$\mu = \exp\left(\int f(x)dx\right),$$

find an expression for $f(x)$. [6]

If instead the integrating factor is $\psi(y)$, a function of y only, what is the expression for ψ? [2]

(c) The equation

$$(3xy^2 + 2y)dx + (2x^2y + x)dy = 0$$

has an integrating factor $\mu(x)$ which depends only on x. Find $\mu(x)$ and hence solve the equation for $y(x)$ explicitly. [6]

15T

(a) A differentiable function of a real variable x is given by $f(x)$. Suppose one is interested in the value of this function when its argument is close to the point $x = a$. State Taylor's theorem explaining how to calculate the value of this function at the point $x = a + \delta$, where δ is sufficiently small. [4]

(b) Write the Maclaurin series for $f(x) = (1+x)^{\frac{1}{3}}$ in the form

$$f(x) = \sum_{n=0}^{\infty} u_n x^n$$

and give the general expression for u_n. [6]

(c) Find the first three non-zero terms in the Maclaurin series for

$$f(x) = \cos\sqrt{\frac{\pi^2}{16} + x}\,.$$

[6]

(d) Find the first three non-zero terms in the approximation valid for large x of $\ln(1 + x + x^2)$. [4]

16R

(a) The vector field $\mathbf{v}$ is given in Cartesian coordinates by

$$\mathbf{v} = (2x(y+z), x^2 - y^2, y^2 - z^2)\,.$$

(i) Calculate $\nabla \cdot \mathbf{v}$. [2]

(ii) A cube with sides of length unity has corners at $(0,0,0), (1,0,0), (0,1,0), (0,0,1)$. Without using the divergence theorem, calculate

$$\int \mathbf{v} \cdot d\mathbf{S}$$

over the surface of the cube, where $d\mathbf{S}$ is the outward-pointing differential element of vector surface area. [6]

(b) By calculating $\nabla \times \mathbf{F}$ in each case, determine which of the following vector fields, given in Cartesian coordinates, are conservative:

$$\mathbf{F}_1 = (3x^2y^2z,\ 2x^3yz,\ x^3y^2)\,,$$
$$\mathbf{F}_2 = (3x^2yz^2,\ 2x^3yz,\ x^3z^2)\,.$$

For each case, if $\mathbf{F}$ is conservative, find a scalar potential ϕ such that $\mathbf{F} = \nabla\phi$. [6]

[*Notation:* $\mathbf{u} \times \mathbf{v}$ *is equivalent to* $\mathbf{u} \wedge \mathbf{v}$.]

For each case, if $\mathbf{F}$ is conservative, explicitly evaluate $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ between the points $A = (0,0,0)$ and $B = (1,1,2)$ along

(i) the straight line from A to B,

(ii) the curve $x = t, y = t, z = 2t^3$ from $t = 0$ to $t = 1$,

and comment on the relation of your results to the values of the associated scalar field ϕ evaluated at A and B. [6]

17Z

(a) Calculate the total differential of the function $p(n, V, T)$, where

$$p \;=\; \frac{nRT}{V} - \frac{n^2 a}{V^2}.$$

[5]

(b) Calculate the gradient of $g(r) = g\left(\sqrt{x^2+y^2+z^2}\right)$, where g is an unspecified differentiable function of one variable. [5]

(c) Let $f(x, y, z) = x^2 + yz$.

(i) Determine the directional derivative of f at the point $(1, 1, -3)$ in the direction $\hat{\mathbf{u}} = \frac{1}{3}(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$. [5]

(ii) Determine the unit vector pointing in the direction in which f increases most rapidly at $(1, 1, -3)$. [5]

18T

A function of a real variable x is given by

$$y = e^{\sin^2 x}.$$

(a) Find all the stationary points of this function and determine whether each of them is a maximum, minimum or point of inflection. [6]

(b) Sketch y as a function of x. [4]

Suppose now that $0 < x < \pi$ and

$$z = \sqrt{\ln y}\,, \qquad (\dagger)$$

(c) Calculate $\dfrac{dz}{dy}$ and $\dfrac{d^2 z}{dy^2}$. [3]

(d) Use the chain rule to calculate $\dfrac{dz}{dx}$ as a function of x. [2]

(e) Use the chain rule to calculate $\dfrac{d^2 z}{dx^2}$ as a function of x. [3]

(f) Substitute for y as a function of x in the equation for z in $(\dagger)$ and calculate $\dfrac{d^2 z}{dx^2}$ directly. [2]

19R*

A string of uniform mass per unit length ρ is stretched along the x-axis under tension T and undergoes small transverse oscillations in the x-y plane with displacement $y(x,t)$. Derive the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

satisfied by $y(x,t)$, where c is a constant which you should determine in terms of ρ and T. [4]

A string of length L is fixed at $x = 0, y = 0$, and at $x = L$ it is attached to a small ring of mass M which is constrained to move without friction on a straight wire parallel to the y-axis. The string undergoes small transverse oscillations in the x-y plane with displacement $y(x,t)$. Show that the equation of motion for the ring is

$$M \frac{\partial^2 y}{\partial t^2}\bigg|_{x=L} = -T \frac{\partial y}{\partial x}\bigg|_{x=L} .$$

[4]

Find the separable solutions to the equation of motion for the string, and show that the allowed frequencies of vibration $\omega_n > 0,\ n = 0, 1, 2, \ldots$, satisfy

$$\cot\left(\frac{\omega_n L}{c}\right) = \frac{M\omega_n}{\rho c} .$$

[4]

Consider now the case $M = 0,\ L = \pi$. Find the displacement, $y(x,t)$, of the string for $t \geqslant 0$ given the initial conditions

$$y(x,0) = 0 , \qquad \frac{\partial y}{\partial t}\bigg|_{t=0} = \sin\frac{1}{2}x + \sin\frac{5}{2}x .$$

[8]

20Z*

(a) Prove the Schwarz inequality

$$\int f^2\,dx \int g^2\,dx \geqslant \left(\int f\,g\,dx\right)^2 .$$

[5]

(b) Determine the derivative with respect to x of

$$\int_{3x}^{x^2} \sin(xt)\,dt.$$

[6]

(c) Let

$$I(a) = \int_0^1 \frac{\sin(a\ln x)}{\ln x}\,dx.$$

Obtain an expression for the derivative of $I(a)$ with respect to a, $I'(a)$. Evaluate this expression and use it to obtain an expression for $I(a)$ and hence determine the value of $I(1)$. [9]

END OF PAPER