NATURAL SCIENCES TRIPOS Part IA

Monday, 9 June, 2014  9:00 am to 12:00 pm

MATHEMATICS (1)

Before you begin read these instructions carefully:

Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

You may submit answers to all of section A, and to no more than five questions from section B.

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on one side of the paper only and begin each answer on a separate sheet. (For this purpose, your section A attempts should be considered as one single answer.)

Questions marked with an asterisk (*) require a knowledge of B course material.

After the end of the examination:

Tie up all of your section A answer in a single bundle, with a completed blue cover sheet.

Each section B question has a number and a letter (for example, 11X). Answers to each question must be tied up in separate bundles and marked (for example 11X, 12T etc) according to the number and letter affixed to each question. Do not join the bundles together. For each bundle, a blue cover sheet must be completed and attached to the bundle, with the correct number and letter written in the box.

A separate green master cover sheet listing all the questions attempted must also be completed. (Your section A answer may be recorded just as A: there is no need to list each individual short question.)

Every cover sheet must bear your examination number and desk number. Calculators are not permitted in this examination.

STATIONERY REQUIREMENTS SPECIAL REQUIREMENTS

6 blue cover sheets and treasury tags  None
Green master cover sheet
Script paper

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.
SECTION A

1

(a) Differentiate

\[
\frac{1}{x^2 + 4}
\]

with respect to \( x \). \([1]\)

(b) Differentiate

\( e^{\sin x} \)

with respect to \( x \). \([1]\)

2

(a) Differentiate \( a^{-x} \) with respect to \( x \), where \( a \) is a constant which satisfies \( a > 0 \) and \( a \neq 1 \). \([1]\)

(b) Evaluate the indefinite integral

\[
\int \frac{\ln(\ln x) dx}{x}
\]

\([1]\)

3

(a) Evaluate the definite integral

\[
\int_0^\pi \tan x \, dx.
\]

\([1]\)

(b) Evaluate the definite integral

\[
\int_{-2}^{-1} \frac{dx}{x}.
\]

\([1]\)
(a) Find the general solution of the differential equation
\[ \frac{dy}{dx} = \cos^2 y \sin x \]
for \(-\pi/2 < y < \pi/2\). [1]

(b) Find the solution of the differential equation
\[ \frac{dy}{dx} = 3y \]
such that \(y = 3\) when \(x = 0\). [1]

5

(a) Find all pairs of coordinates where the curve defined by
\[ 7x^2 - y^2 = 7 \]
meets the straight line
\[ y = x + 1. \] [1]

(b) Sketch the curve defined by the equation \((x - 1)^2 + 2y^2 = 3\). [1]

6

(a) A sphere has radius 10 m. Drawn on its surface is a circular patch which has area 10 m\(^2\). What fraction of the surface of the sphere does this cover? [1]

(b) A square pyramid has all of its sides of length 1 m. What is its volume? [1]
(a) Sketch the graph of
\[ y = \frac{1}{1 + \tan x}. \]

(b) Sketch the graph of
\[ y = e^{-x^3}. \]

(a) Find the values of \( x \) at the stationary points of the function
\[ y = x^3 - 2x^2 - 7x + 6. \]

(b) The function \( y = x^3 - 3x + 7 \) has a stationary point at \( x = 1 \). Is this point a maximum, minimum or a point of inflection?

(a) What is the area bounded by the curve \( y = x^2 - 3x + 2 \), the positive half of the \( x \)-axis, the positive half of the \( y \)-axis and the line \( x = 1/2 \)?

(b) Sketch the curve(s) defined by the relation
\[ y^2 = x^3. \]
(a) The straight line $L$ is defined by the equation $y = 2x + 3$. Find the equation of the line $L'$ that is perpendicular to $L$ and intersects $L$ where $L$ crosses the $x$-axis. \[1\]

(b) Express \[
\frac{13(x + 1)}{(x - 4)(x + 9)}
\] as a sum of partial fractions. \[1\]
(a) Calculate Det $A$ and Tr $A$ where

$$A = \begin{pmatrix} 1 & -4 & 7 \\ -4 & 4 & -4 \\ 7 & -4 & 1 \end{pmatrix}.$$ 

From the value of Det $A$ make a deduction about the eigenvalues of $A$. [4]

(b) Calculate the eigenvalues and the corresponding normalised eigenvectors of $A$. Verify that the eigenvectors are mutually orthogonal. [10]

(c) By expressing an arbitrary vector $r$ in terms of the eigenvectors or otherwise, show that a non-zero vector $e$ exists such that

$$Ar \cdot e = 0$$

for all $r$. [4]

(d) Describe in words the action of $A$ on an arbitrary non-zero vector. [2]

12T

(a) Express the cube roots of $i - 1$ in terms of their modulus and argument. [4]

(b) Find all the solutions to the equation

$$\tanh z = -i.$$ [5]

(c) Given that $z = 2 + i$ solves the equation

$$z^3 - (4 + 2i)z^2 + (4 + 5i)z - (1 + 3i) = 0,$$

find the remaining solutions. [6]

(d) Use complex numbers to show that

$$\cos 4\theta = 8 \cos^4 \theta - 8 \cos^2 \theta + 1.$$ [5]
(a) Solve the following differential equations for \( y(x) \) subject to the listed boundary conditions, making your answer explicit for \( y \).

(i) \( \frac{dy}{dx} + 3y = 8 \), with \( y(0) = 4 \). \[3\]
(ii) \( \frac{dy}{dx} - y \cos x = \frac{1}{2} \sin 2x \), with \( y(0) = 0 \). \[7\]

(b) The function \( y(x) \) satisfies the differential equation

\[
\frac{d^2y}{dx^2} + 7 \frac{dy}{dx} + 12y = 2e^{-3x}.
\]

Solve the equation for \( y(x) \) subject to \( y(0) = 1 \) and \( \frac{dy}{dx} \bigg|_{x=0} = 0 \). \[10\]

14Z

(a) An arbitrary point along a straight line in a three-dimensional space can be written as \( \mathbf{r}_1 = \mathbf{a} + \lambda \mathbf{b} \), where \( \lambda \) is a scalar parameter and \( \mathbf{b} \) is a unit vector. Obtain a formula for the minimum distance between \( \mathbf{r}_1 \) and \( \mathbf{r}_2 = \mathbf{c} + \mu \mathbf{d} \), where \( \mathbf{d} \) is a unit vector, assuming that the two lines are not parallel. \[5\]

(b) Find all vectors \( \mathbf{x} \) that obey the equation \( \mathbf{x} \cdot \mathbf{p} = k \), where \( \mathbf{p} \) is a fixed non-zero vector in a three-dimensional space and \( k \) is a fixed scalar.

[Hint: Your answer should contain an arbitrary non-zero vector \( \mathbf{q} \) which can be taken to be non-collinear with \( \mathbf{p} \), i.e., \( \mathbf{p} \times \mathbf{q} \neq 0 \). You should treat the cases \( \mathbf{p} \cdot \mathbf{q} \neq 0 \) and \( \mathbf{p} \cdot \mathbf{q} = 0 \) separately.]

[Notation: \( \mathbf{u} \times \mathbf{v} \) is equivalent to \( \mathbf{u} \wedge \mathbf{v} \).] \[7\]

(c) A particle moves along a path on which the position coordinate in terms of a parameter \( t \) is given by

\[
x = \frac{\cos t}{\sqrt{1 + t^2}}, \quad y = \frac{\sin t}{\sqrt{1 + t^2}}, \quad z = \frac{t}{\sqrt{1 + t^2}}.
\]

Express the equation for a point on the path in spherical polar coordinates \( r(t), \theta(t), \phi(t) \). \[8\]
(a) The area of integration, $D$, is defined in plane polar coordinates $(r, \phi)$ by the inequality $r_2 \leq r \leq r_1$, where $r_1 = 1 + \cos \phi$ and $r_2 = 3/2$.

(i) Sketch the area of integration. [4]

(ii) Calculate the value of the area $D$. [6]

(iii) Evaluate the following integral over this area:

$$\iint_D \frac{x + y + xy}{x^2 + y^2} \, dx \, dy.$$ [5]

(b) Evaluate the triple integral:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left[ -\frac{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}{a^2} \right] \, dx \, dy \, dz,$$

where $a > 0$, $x_0$, $y_0$ and $z_0$ are real constants. [5]

16Z

(a) A function $f$ of two variables $x$ and $y$ is defined as

$$f(x, y) = 2x^3 + 6xy^2 - 3y^3 - 150x.$$

Determine the positions of the stationary points of $f$ and their characters (maximum, minimum or saddle point). [8]

(b) A function $g$ of two variables $x$ and $y$ is defined as

$$g(x, y) = x^4 + y^4 - 36xy.$$

Sketch the contours of $g$ in the $x$-$y$ plane, indicating on the sketch the positions and characters of all the stationary points. [12]
(a) Suppose \( f(x) \) is a periodic function with period \( 2\pi \). Write down its Fourier series and give expressions for the coefficients that appear in it. \[3\]

(b) The function \( g(x) = e^x \) is defined on the interval \( -\pi \leq x < \pi \). Sketch the periodic continuation of \( g(x) \) with period \( 2\pi \), between \( x = -3\pi \) and \( x = 3\pi \). If we were to calculate the Fourier series of this periodic continuation of \( g(x) \), what value would it take at the point \( x = \pi \)? \[4\]

(c) Consider the function \( h(x) = x(\pi - x) \) defined on the interval \( 0 \leq x < \pi \). By considering the appropriate periodic continuation of \( h(x) \) over the real line, show that the half-range sine series for \( h(x) \) is

\[
h(x) = \frac{8}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \sin[(2n+1)x].
\]

Hence demonstrate that

\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}.
\]

\[10\]
(a) Suppose $X$ is a discrete random variable that takes the integer values $0, 1, 2, \ldots, N$. Its normalised probability distribution is denoted by $P(X)$.

Write down expressions for the mean $\mu$, the variance $\sigma^2$, and the probability $P(X < Y)$, where $Y$ is a fixed positive integer less than $N$. [3]

(b) A pond contains $K$ trout and $N - K$ carp. Bruce goes fishing at the pond and in one day catches $M$ fish. Note that Bruce never returns a fish to the pond once it is caught.

Calculate the number of ways $M$ fish (of any species) can be caught from a pond of $N$ fish. Suppose Bruce catches $X$ trout out of his haul of $M$ fish. Show that the number of ways of catching $X$ trout out of the haul of $M$ fish is

$$\binom{K}{X} \binom{N - K}{M - X}.$$  

Assuming that trout and carp are equally likely to be caught, show that the probability that Bruce catches $X$ trout in one day is

$$P(X) = \binom{K}{X} \binom{N - K}{M - X} / \binom{N}{M}.$$  [5]

(c) Suppose the pond contains 2 trout and 8 carp, and Bruce catches 2 fish in total. What is the probability that of these two fish (i) none are trout, (ii) one is a trout, and (iii) both are trout? Verify that the three probabilities sum to 1. Consequently, determine the mean and variance of the probability distribution $P(X)$ for this case. (You may leave your answers in reduced fractional form.) [10]

[Recall that $\binom{N}{n} = N!/(n!(N - n)!)$.]  

(d) The next day Bruce goes fishing at a large lake that contains only trout and carp but in enormous quantities, i.e. both $K$ and $N - K$ are much larger than $M$. In this limit $P(X)$ approaches the binomial distribution. Give a qualitative explanation for why this is so. [2]
19Y*

(a) A differentiable function $f(x)$ is expanded using the Maclaurin series. Derive an expression which determines the interval for $x$ within which the series is absolutely convergent.

Use your result to determine the interval for $x$ within which the Maclaurin series for $f(x) = \ln(2 + x)$ converges absolutely. Determine whether or not the series converges at the end points of the interval.

(b) Find the first three terms in the Maclaurin series for

$$f(x) = e^{-x}(1 + x)^{-1/2}.$$

(c) Establish the convergence or divergence of the series $\sum_{n=1}^{\infty} u_n$ whose $n$-th terms, $u_n$, are

(i) $\frac{2^n}{n \ln n}$,

(ii) $\frac{\sin n}{n^2}$.

(d) Sum the series

$$S(x) = \frac{x^4}{3(0!)} + \frac{x^5}{4(1!)} + \frac{x^6}{5(2!)} + \ldots.$$
20R*

The point \((a, b)\) is a stationary point of the function \(f(x, y)\) subject to the constraint \(g(x, y) = 0\). Using the method of Lagrange multipliers show that

\[
\begin{vmatrix}
\frac{\partial f}{\partial x}(a, b) & \frac{\partial g}{\partial x}(a, b) \\
\frac{\partial f}{\partial y}(a, b) & \frac{\partial g}{\partial y}(a, b)
\end{vmatrix} = 0.
\]

(a) By considering the function \(f(x, y) = x^2 + y^2\), use the method of Lagrange multipliers to find the maximum distance from the origin to the curve

\[x^2 + y^2 + xy - 4 = 0.\]

(b) In a school, two horizontal corridors, \(0 \leq x \leq a, \ y \geq 0\) and \(x \geq 0, \ 0 \leq y \leq b\) meet at right angles. The caretaker wishes to know the maximum possible length, \(L\), of a ladder that may be carried horizontally around the corner. Regarding the ladder as a stick, use the method of Lagrange multipliers to calculate \(L\) by first placing the ends of the ladder at the points \((a + X, 0)\) and \((0, b + Y)\) and imposing the condition that the corner \((a, b)\) be on the ladder. Then show that at the constrained stationary point, the value of \(X\) satisfies the equation

\[(X^3 - ab^2)(X + a) = 0,\]

and hence show that \(L = (a^{2/3} + b^{2/3})^{3/2}\). 

END OF PAPER