UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS **Part IB & II (General)**

Friday, 31 May, 2013 9:00 am to 12:00 pm

MATHEMATICS (2)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, ***6A****).*

Answers must be tied up in ***separate*** *bundles, marked* ***A, B or C*** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A ***separate*** *green master cover sheet listing all the questions attempted* ***must*** *also be completed.*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Green master cover sheet
Script paper

SPECIAL REQUIREMENTS
Calculator - students are permitted to bring an approved calculator.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1B

Consider the equation

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2 y = 0 \qquad (*)$$

on the domain $-1 \leqslant x \leqslant 1$ with n an integer. We require y and its derivative to be bounded at $x = \pm 1$.

(i) Put this equation into Sturm-Liouville form and identify the weight function. [3]

(ii) Let y_n and y_m be two eigenfunctions of $(*)$ with associated eigenvalues n^2 and m^2.

State and prove the orthogonality property for these eigenfunctions, assuming $n \neq m$. [4]

Go on to prove that

$$\int_{-1}^{1} \frac{dy_n}{dx}\frac{dy_m}{dx}\sqrt{1-x^2}\,dx = 0\,.$$

[3]

(iii) Demonstrate that $y_n = \cos\left[n \cos^{-1} x\right]$ is an eigenfunction of $(*)$. [3]

Show, using de Moivre's theorem or otherwise, that y_n is a polynomial in x of degree n. [4]

Compute the first three polynomials, y_0, y_1, and y_2, and verify that they are orthogonal. [3]

2B

The free decay of the Earth's axisymmetric magnetic field can be modelled by the equations

$$\nabla^2 B + s\,B = 0, \qquad \text{when} \quad r < R,$$
$$\nabla^2 B = 0, \qquad \text{when} \quad r > R.$$

Here R is the radius of the Earth's spherical core and s is the decay rate. We require that $B \to 0$ as $r \to \infty$, and that B and $\partial B/\partial r$ are continuous at $r = R$.

[*Hint: The axisymmetric spherical Laplacian is*

$$\nabla^2 B = \frac{1}{r^2}\,\frac{\partial}{\partial r}\left[r^2 \frac{\partial B}{\partial r}\right] + \frac{1}{r^2 \sin\theta}\,\frac{\partial}{\partial \theta}\left[\sin\theta \frac{\partial B}{\partial \theta}\right]\,. \quad]$$

(i) Using separation of variables, find an expression for B as an expansion in Legendre polynomials $P_l(\cos\theta)$ when $r > R$.

[*Hint: Recall that*

$$\frac{d}{d\mu}\left[(1-\mu^2)\,\frac{dy}{d\mu}\right] + \lambda y = 0$$

only admits regular solutions at $\mu = \pm 1$ *when* $\lambda = l(l+1)$ *and these are the* $P_l(\mu)$.] [6]

(ii) Show that when $r < R$, B can be expressed as

$$B(r,\,\theta) = \sum_{l=0}^{\infty} B_l\, f_l(s^{1/2}\,r)\, P_l(\cos\theta)\,,$$

where B_l is a constant and $f_l(s^{1/2}\,r)$ satisfies the equation

$$r^2\,\frac{d^2 f_l}{dr^2} + 2r\,\frac{df_l}{dr} + (sr^2 - l(l+1))f_l = 0\,.$$

[*Hint: Note that the second solution to this equation is singular at* $r = 0$ *but* f_l *is bounded at* $r = 0$.] [6]

(iii) Impose the two boundary conditions at $r = R$ to obtain the following eigenvalue equation

$$f_l(s^{1/2}\,R) = 0\,,$$

using the identity $x f'_{\nu+1}(x) + (\nu+2) f_{\nu+1}(x) = x f_\nu(x)$. [6]

(iv) The smallest zero of $f_l(x)$ is π. Hence write down an expression for the smallest value of s in terms of R. [2]

3B

(i) Verify that

$$H(\mathbf{r}, \mathbf{r}') = -\frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|}$$

is a solution to the equation

$$\nabla^2 H = \delta(\mathbf{r} - \mathbf{r}'),$$

in three-dimensions. [7]

Hence write down the general solution for the gravitational potential Φ satisfying Poisson's equation

$$\nabla^2 \Phi = 4\pi G\rho,$$

where G is the gravitational constant and $\rho = \rho(\mathbf{r})$ is a general mass distribution. What is the potential Φ associated with the point mass $\rho = M\delta(\mathbf{r})$? [3]

(ii) Consider the gravitational potential associated with a spherical planet of radius R and constant mass density ρ.

Show that the Green's function of the Laplacian may be written as

$$H(\mathbf{r},\, \mathbf{r}') = -\frac{1}{4\pi\sqrt{r'^2 - 2r'\, r\, \cos\theta' + r^2}}$$

where θ' is the angle between $\mathbf{r}$ and $\mathbf{r}'$, $r = |\mathbf{r}|$; $r' = |\mathbf{r}'|$. [3]

Insert this expression in the Green's function solution for Φ to obtain

$$\Phi = -2\pi G\rho \int_0^R \frac{r'}{r}\left(r' + r - |r' - r|\right)\, dr'\,.$$

[*Hint: Use spherical coordinates where the z' axis points in the same direction as* $\mathbf{r}$.] [4]

Perform the final r' integration to obtain the gravitational potential for $r < R$. [3]

4B

(i) Use Cauchy's residue theorem to evaluate the complex integral

$$\int_C \frac{f(z)}{z - z_0}\, dz,$$

where C is a closed contour in the complex plane, the point z_0 lies within the region enclosed by C, and f is analytic in this region. [2]

(ii) Consider the contour integral

$$\int_C \frac{e^{iz}}{a^2 + z^2} dz\,, \qquad (*)$$

where $a > 0$ and C is the closed contour consisting of the real axis between $-R$ and R and a semicircle in the upper half plane of radius R connecting the two points $(-R, 0)$ and $(R, 0)$. The sense of the integration path is counterclockwise and $R > a$.

Locate the integrand's singularities and then evaluate the integral with the residue theorem. [5]

By considering the real part of $(*)$ deduce that

$$\int_{-\infty}^{\infty} \frac{\cos x}{a^2 + x^2}\, dx = \frac{\pi}{a} e^{-a}\,.$$

[3]

(iii) Consider the real integral

$$\int_0^{2\pi} \frac{1}{1 + \epsilon \cos\theta}\, d\theta\,, \qquad (**)$$

for $-1 < \epsilon < 1$.

Turn this integral into a closed contour integral and specify the integration path. [2]

Show that the integrand possesses two simple poles in the complex z plane

$$z_{\pm} = \frac{-1 \pm \sqrt{1 - \epsilon^2}}{\epsilon}\,,$$

and that one is enclosed by the integration path. [5]

Use the residue theorem to calculate $(**)$. [3]

5C

(i) Calculate the Fourier transform of the triangle function

$$f(x) = \begin{cases} 1 - |x|, & |x| < 1, \\ 0, & \text{otherwise.} \end{cases} \qquad (*)$$

[5]

The Fourier transform with respect to x of a function $u(x,t)$ is given by

$$\tilde{u}(k,t) = \int_{-\infty}^{\infty} u(x,t) e^{-ikx} dx \,.$$

(ii) Using the formal limit definition of a derivative, derive expressions for the Fourier transforms with respect to x of $\dfrac{\partial u}{\partial t}$ and $\dfrac{\partial u}{\partial x}$. [*Hint: You may assume that* $u \to 0$ *as* $|x| \to \infty$.] [5]

(iii) If $u(x,t)$ satisfies

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \,,$$

write down the ordinary differential equation obeyed by the Fourier transform $\tilde{u}(k,t)$ of $u(x,t)$. [4]

(iv) Find $u(x,t)$ subject to the following initial conditions at $t = 0$

$$u = f(x) \,, \quad \frac{\partial u}{\partial t} = 0 \,,$$

where $f(x)$ is the triangle function $(*)$. Assume again that $u \to 0$ as $|x| \to \infty$. [6]

6C

(i) State the transformation rules for tensors of rank one and two. [2]

(ii) If u_i and v_j are rank one tensors (i.e. vectors), show that $u_i v_j$ is a rank two tensor. [2]

(iii) Consider a rank two tensor B_{ij}. Let S_{ij} and A_{ij} be symmetric and anti-symmetric rank two tensors where $B_{ij} = S_{ij} + A_{ij}$. Write S_{ij} and A_{ij} in terms of the components of B_{ij}. [2]

(iv) Show that if R_{ij} is a symmetric rank two tensor, then

$$R_{ij} B_{ij} = R_{ij} S_{ij} \,,$$

where S_{ij} is the symmetric part of B_{ij} defined above. [2]

(v) Show that the *tensor product* of a rank two tensor with a vector is a rank three tensor. [2]

(vi) Let S_{ijk} be a rank three tensor. A contraction of S_{ijk} is defined as

$$C_k = \sum_i S_{iik} \,.$$

Show that C_k is a vector. [4]

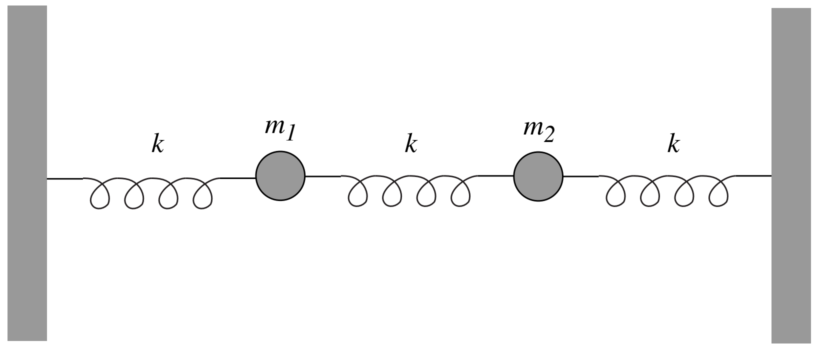
(vii) Using suffix notation and the *Levi-Civita* pseudo-tensor, ϵ_{ijk}, prove the following vector identity

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}\nabla \cdot \mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} - \mathbf{B}\nabla \cdot \mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \,.$$

[6]

7C

Two objects with masses m_1 and m_2 are connected to two rigid walls by three springs with identical spring constants k, as sketched below. Let x_1 and x_2 be the displacements of m_1 and m_2 from their equilibrium positions, respectively. The motion of the objects is confined to the horizontal (x) direction.



(i) Find the normal modes of oscillation and their associated frequencies. [10]

(ii) At $t = 0$, the masses are each displaced from their equilibrium position by a distance x_0 and away from each other, then released from a state of rest. Solve for x_1 and x_2 and express them as linear combinations of the normal modes. [5]

(iii) If the masses are initially at the equilibrium position, but m_2 is given an initial velocity u_0, while m_1 is initially at rest, solve for x_1 and x_2. Describe this motion in terms of the normal modes. [5]

8B

(i) Consider the set of functions of x

$$\mathcal{F} = \left\{ x,\ -x,\ \frac{1}{x},\ -\frac{1}{x} \right\}$$

endowed with the operation of functional composition, i.e. if $f, g \in \mathcal{F}$ then $f \bullet g = f(g(x))$.

Prove that $\mathcal{F}$ is a group. Construct a 'multiplication' table as part of your answer. [6]

What are the subgroups of $\mathcal{F}$? [2]

Prove that $\mathcal{F}$ is isomorphic to the dihedral group D_2. [2]

(ii) State Lagrange's theorem. Subsequently show that if the order of a group is prime then that group has no proper subgroups. [3]

(iii) Define the order of a group element. Prove that the order of any group element is a factor of the group's order. [3]

(iv) Show that if the order of a group is prime then that group is cyclic. [2]

(v) Suppose $\mathcal{G}$ is a cyclic group but not of prime order. Demonstrate that $\mathcal{G}$ contains a proper cyclic subgroup. [2]

9B

(i) What is meant by the terms *normal subgroup* and *group homomorphism*? [2]

(ii) Consider the group homomorphism $\phi : \mathcal{G}_1 \to \mathcal{G}_2$.

Prove that the image of ϕ is a subgroup of $\mathcal{G}_2$. [2]

Prove that the kernel K of ϕ is a *normal* subgroup of $\mathcal{G}_1$. [3]

Demonstrate that if K consists only of the identity element then ϕ is injective (i.e. one-to-one). [3]

(iii) If D_3 is the symmetry group of the equilateral triangle, describe the geometrical action of the six members of D_3 and give the minimal generating set of the group. Express the members of D_3 in terms of the generators. [6]

Identify the members of D_3 with the permutation group Σ_3. Hence show that the two groups are isomorphic. [4]

10B

(i) Define *faithful representation*, *equivalent representations*, *irreducible representation*, and the *character* of a representation. [4]

(ii) Let $\mathcal{G}$ be the following set of real 4×4 matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Show that $\mathcal{G}$ is a group under the operation of matrix multiplication.Go on to show that it is a *faithful representation* of the dihedral group D_2. [5]

By finding an invariant subspace of the representation, prove that $\mathcal{G}$ is reducible. [3]

(iii) What are the conjugacy classes of Z_n, the cyclic group of order n? What can you say about the number and dimensions of its irreducible representations?

[*Hint: You may need the result* $|\mathcal{G}| = \sum_{k=1}^{n_p} d_k^2$, *where* n_p *is the number of irreducible representations of any group* $\mathcal{G}$ *and* d_k *is the dimension of the* k*'th representation.*] [4]

Give the irreducible representations of Z_n. Write down the associated character table for the special case $n = 3$. [4]

END OF PAPER