UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS   Part IB & II (General)

Tuesday, 28 May, 2013   9:00 am to 12:00 pm

MATHEMATICS (1)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, ***6A****).*

Answers must be tied up in **separate** *bundles, marked* ***A, B or C*** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A ***separate*** *green master cover sheet listing all the questions attempted* ***must*** *also be completed.*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Green master cover sheet
Script paper

SPECIAL REQUIREMENTS
Calculator - students are permitted to bring an approved calculator.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1C

(i) Using Cartesian coordinates, show that for arbitrary vector fields $\mathbf{A}(x, y, z)$ and $\mathbf{B}(x, y, z)$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})\,.$$

[6]

(ii) State the divergence theorem, and use it to show that for a scalar field $a(x, y, z)$ and vector field $\mathbf{B}(x, y, z)$

$$\int\int\int_V \nabla a \cdot (\nabla \times \mathbf{B})\, dV = -\int\int_S (\nabla a \times \mathbf{B}) \cdot \hat{\mathbf{n}}\, dS\,, \qquad (*)$$

where V is a given volume, and $\hat{\mathbf{n}}$ is the unit vector outward normal to its surface, S. [6]

(iii) Consider the particular case $a = xy + z^2$ and $\mathbf{B} = y\mathbf{i} - yz\mathbf{j} + x\mathbf{k}$, for Cartesian unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.
Verify both sides of $(*)$, where V is a circular cylinder of height h and radius 1, with base $x^2 + y^2 = 1$ at $z = 0$. [8]

2C

A damped wave on a string can be described by the equation

$$u_{tt} = c^2 u_{xx} - \alpha u_t \,,$$

where subscripts denote partial derivatives and α and c are constants.

(i) Use the method of *separation of variables* to find two ordinary differential equations. [4]

(ii) Consider a string between $-L \leqslant x \leqslant L$ with fixed endpoints $u(x = -L) = u(x = L) = 0$. If the string is plucked in the centre, we might expect the solutions to be symmetric about $x = 0$. Show that the general solution for *symmetric disturbances* can be written in the following form

$$u(x,t) = \sum_{n=1}^{\infty} e^{-\alpha t/2} \cos\left(\frac{n\pi x}{2L}\right) \mathrm{Re}\left[A_n e^{i\omega_n t} + B_n e^{-i\omega_n t}\right] \,, \qquad (*)$$

where n is an *odd integer* and Re denotes real part. [6]

(iii) Give an expression for ω_n as a function of α, n, L and c. How small must the damping coefficient, α, be for oscillatory solutions to exist? Describe what happens if $\alpha < 0$. [3]

(iv) If the string is plucked so that at $t = 0$

$$\frac{\partial u}{\partial t} = 0, \quad \text{and} \quad u(x, t = 0) = e^{-|x|/l} \,,$$

find the coefficients A_n and B_n in $(*)$. How do the coefficients simplify in the limit when $l \ll L$, as required to impose $u(x = -L) = u(x = L) = 0$? [7]

3C

(i) Find the general solution $y(x)$ to the homogeneous second-order linear differential equation

$$\frac{d^2y}{dx^2} - \frac{1+x}{x}\frac{dy}{dx} + \frac{y}{x} = 0\,.$$

[6]

[*Hint: Look for a particular solution of the form* $y_p(x) = g(x)e^x$.]

(ii) Find the Green's function for this equation in the region $-1 \leqslant x \leqslant 1$, subject to the homogeneous boundary conditions $y(-1) = 0$ and $y(1) = 0$. [8]

(iii) Use the Green's function found above to solve the inhomogeneous differential equation

$$\frac{d^2y}{dx^2} - \frac{1+x}{x}\frac{dy}{dx} + \frac{y}{x} = x\,,$$

subject to the same boundary conditions. [6]

4C

The Fourier transform $\widetilde{f}(k)$ of a function $f(x)$ is defined by

$$\widetilde{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x) dx \,,$$

and the correlation $h(x)$ between two functions $f(x)$ and $g(x)$ is defined by

$$h(x) = \int_{-\infty}^{\infty} (f(y))^* \, g(x+y) dy \,,$$

where $*$ denotes a complex conjugate.

(i) Prove that

$$\widetilde{h}(k) = \left(\widetilde{f}(k)\right)^* \widetilde{g}(k) \,.$$

[6]

(ii) Use this result to prove *Parseval's theorem*

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\widetilde{f}(k)|^2 dk \,.$$

[6]

(iii) Verify Parseval's theorem for the following function

$$f(x) = \begin{cases} 1 & |x| \leqslant 1 \,, \\ 0 & |x| > 1 \,. \end{cases}$$

[8]

[*Hint:* $\displaystyle\int_0^{\infty} \frac{\sin x \cos x}{x} dx = \pi/4$.]

5A

(i) If M is an invertible complex matrix with Hermitian conjugate $M^\dagger$ and inverse M^{-1} show that

$$(M^\dagger)^{-1} = (M^{-1})^\dagger \,.$$

[2]

(ii) If A is an anti-Hermitian matrix, i.e. one such that $A^\dagger = -A$, show, by diagonalizing iA, that

$$|\det(1 + A)|^2 \geqslant 1 \,,$$

and hence that $1 + A$ is always invertible. [6]

(iii) If A is an anti-Hermitian matrix, show that

$$U = (1 - A)(1 + A)^{-1} \qquad (*)$$

is a unitary matrix, that is $U^\dagger = U^{-1}$. [6]

(iv) If U is a unitary matrix such that $1 + U$ is invertible, show that there is a unique matrix A satisfying $(*)$. Show that the matrix A is indeed anti-Hermitian. Give an example of a unitary matrix for which $1 + U$ is not invertible. [6]

6A

(i) If M is an anti-symmetric $n \times n$ matrix show that

$$\det M = (-1)^n \det M \,,$$

and hence if n is odd, $\det M$ must vanish. [2]

(ii) If M is a real anti-symmetric $n \times n$ matrix show that M^2 is a real symmetric non-positive matrix, i.e.

$$\mathbf{x}^T M^2 \mathbf{x} \leqslant 0$$

for all vectors $\mathbf{x}$, where ^{T} denotes transpose. Hence show that if n is odd then M^2 must have at least one vanishing eigenvalue. [3]

(iii) If $\mathbf{e}_1$ is an eigenvector of M^2 with non-vanishing eigenvalue $\lambda_1 = -\mu_1^2$, with $\mu_1 > 0$, show that $\mathbf{e}_2 = M\mathbf{e}_1$ is also an eigenvector of M^2, orthogonal to $\mathbf{e}_1$ with the same eigenvalue. [5]

(iv) By considering the remaining eigenvectors, $\mathbf{e}_3, \ldots, \mathbf{e}_n$, conclude that the non-vanishing eigenvalues of M^2 occur in, not necessarily distinct, pairs. [4]

(v) Hence show, using the basis of eigenvectors of M^2 , that the original matrix M may be cast in block diagonal form with each block being either 2×2 anti-symmetric with entries $\pm\mu_1$, $\pm\mu_2$, $\ldots$ or a block with zero entries. [6]

7A

(i) Write down the Cauchy Riemann equations for the real and imaginary parts, u, v of the analytic function $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$ and hence show that the level sets, $u =$ constant and $v =$ constant, are orthogonal, and that $|\nabla u| = |\nabla v|$. [3]

(ii) Show that u satisfies Laplace's equation $\nabla^2 u = (\partial_x^2 + \partial_y^2)u = 0$, where $\partial_x = \frac{\partial}{\partial x}$, $\partial_y = \frac{\partial}{\partial y}$. [2]

(iii) Using the analytic function $f(z) = \cosh^{-1} z$, show that the level sets $u =$ constant and $v =$ constant form an orthogonal system of ellipses and hyperbolae. [5]

(iv) Hence show that $\phi = u - \cosh^{-1}(\sqrt{2})$ is a solution of Laplace's equation which vanishes on the ellipse

$$\frac{x^2}{2} + y^2 = 1\,.$$

How does ϕ behave as $x, y \to \infty$? [5]

(v) If

$$F(z, \bar{z}) = \bar{z}H(z) + G(z) = U(x, y) + iV(x, y)\,,$$

where $H(z)$ and $G(z)$ are analytic functions of z and $\bar{z} = x - iy$, show that U and V satisfy the fourth order partial differential equations

$$\nabla^4 U = (\partial_x^2 + \partial_y^2)(\partial_x^2 + \partial_y^2)U = 0$$
$$\nabla^4 V = (\partial_x^2 + \partial_y^2)(\partial_x^2 + \partial_y^2)V = 0 \quad .$$

[5]

8A

(i) Find a series solution of the differential equation

$$(1 - x^3)y'' - 6x^2y' - 6xy = 0$$

subject to the boundary conditions $y(0) = 1\,, y'(0) = 0$. [5]

(ii) Sum the series and verify that the sum satisfies the differential equation. [5]

(iii) If $P_n(x)$ is a Legendre Polynomial, that is a polynomial of degree n satisfying Legendre's equation

$$\frac{d}{dx}\Big((1 - x^2)\frac{dy}{dx}\Big) + n(n+1)y = 0\,,$$

find the equation satisfied by $v(x)$ if $y = v(x)P_n(x)$ is a solution of Legendre's equation. [3]

(iv) Give the general solution of your equation in terms of an explicit integral. [2]

(v) Hence show that any solution of Legendre's equation which is linearly independent of $P_n(x)$ must behave like a logarithm of $1 \pm x$ near $x = \mp 1$. [3]

(vi) How do those solutions of Legendre's equation which are bounded as $|x| \to \infty$ behave as $|x| \to \infty$? [2]

9A

(i) Write down the Euler-Lagrange equations governing the stationary values of the functional

$$I[y(x)] = \int_a^b F(y, y', x)dx$$

among functions whose endpoint values $y(a)$ and $y(b)$ are fixed. [2]

(ii) Derive first integrals of the Euler-Lagrange equations in the cases

(a) the integrand F has no explicit dependence on y, $F = F(y', x)$, [1]

(b) the integrand F has no explicit dependence on x, $F = F(y, y')$. [3]

(iii) Suppose

$$F = y\sqrt{1 + (y')^2} - \lambda y \,,$$

obtain a first integral . [2]

(iv) If $y' = \tan\psi$, and assuming that a solution exists for $y \geqslant 0$ with a maximum at which $\psi = 0, y = y_0$ and $y_0 > 0$, find an expression for λ in terms of ψ, y, y_0 with $y_0 > y$. [4]

Hence show that for solutions of this type $\lambda > 1$. [2]

(v) Show that if $\psi = \alpha$ at $y = y_1$, where $y_0 > y_1$, then for $y_0 > y > y_1$,

$$\sin^2(\frac{\psi}{2}) = \frac{y_1}{y}\,\frac{y_0 - y}{y_0 - y_1}\sin^2(\frac{\alpha}{2})\,.$$

[6]

10A

The vertical displacement of the skin of a drum with circular cross section and radius a satisfies

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0\,.$$

(i) If $u = e^{i\omega t}R(r)$, where r, θ are plane polar coordinates, find an ordinary differential equation satisfied by $R(r)$ and show that it is in self-adjoint form with a certain weight function which should be specifed. You may assume that in plane polar coordinates

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\big(r\frac{\partial u}{\partial r}\big) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\,.$$

[4]

(ii) Show that the boundary condition $u = 0$ at $r = a$ defines an eigenfunction problem, with real and positive eigenvalues λ such that the frequencies $\nu = \frac{\omega}{2\pi}$ are real. [4]

(iii) Show that the eigenfunctions with distinct eigenvalues are orthogonal with respect to a suitable inner product which should be specified. [4]

(iv) Obtain an upper bound for the lowest non-vanishing frequency ν, using the trial function $f(r) = (1 - (\frac{r}{a})^p)$ and picking the constant p so as to give the best possible bound. [8]

END OF PAPER