UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Wednesday, 12 June, 2013 9:00 am to 12:00 pm

MATHEMATICS (2)

Before you begin read these instructions carefully:

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

You may submit answers to ***all*** *of section A, and to no more than* ***five*** *questions from section B.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on one side of the paper only and begin each answer on a separate sheet. *(For this purpose, your section A attempts should be considered as one single answer.)*

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Tie up ***all of your section A answer*** *in a single bundle, with a completed blue cover sheet.*

Each section B question has a number and a letter (for example, ***11S****). Answers to each question must be tied up in* ***separate*** *bundles and marked (for example,* ***11S, 12W*** *etc) according to the number and letter affixed to each question.* ***Do not join the bundles together****. For each bundle, a blue cover sheet* ***must*** *be completed and attached to the bundle, with the correct number and letter written in the box.*

A ***separate*** *green master cover sheet listing all the questions attempted* ***must*** *also be completed. (Your section A answer may be recorded just as A: there is no need to list each individual short question.)*

Every cover sheet must bear your examination number and desk number.
Calculators are not permitted in this examination.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Green master cover sheet
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1

Given vectors $\boldsymbol{a} = (1, 1, -2)$, $\boldsymbol{b} = (2, -1, 3)$ and $\boldsymbol{c} = (0, 1, -1)$ in Cartesian coordinates, find

(a) $(\boldsymbol{a} \cdot \boldsymbol{b})\ \boldsymbol{c} + \boldsymbol{a}\ (\boldsymbol{b} \cdot \boldsymbol{c})$, [1]

(b) $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$. [1]

2

(a) Evaluate $\cosh^2 x - \sinh^2 x$. [1]

(b) Given that $\tanh x = u$, find an expression for $\sinh x$ in terms of u. [1]

3

Find the real and imaginary parts of the complex number

$$\frac{2 - 3i}{2 + 3i}\ .$$

[2]

4

Find the rate of change (directional derivative) of $\phi(x, y, z) = x^2 y + xz$ at the point $(1, 2, -1)$ in the direction of the vector $\boldsymbol{s} = (2, 0, 1)$. [2]

5

Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\ .$$

[2]

6

Find the first two non-zero terms of the Taylor series expansion of the function $f(x) = \cos(x)\sin(x)$ about the point $x = \pi/4$. [2]

7

(a) Find the spherical polar coordinates (r, θ, ϕ) of a point P whose Cartesian coordinates are $(x, y, z) = \left(\sqrt{3}/4, 3/4, 3/2\right)$. [1]

(b) Identify on a sketch the polar and the azimuthal angles of P. [1]

8

(a) Calculate $\dfrac{\partial^2 F}{\partial x \partial y}$, where $F(x, y) = \exp[\sin(xy)]$. [1]

(b) Calculate the divergence of the vector field $\boldsymbol{G}(x, y, z) = (x^2 z,\ y^3 z,\ x^2 - y^2)$. [1]

9

Let $\boldsymbol{r} = (x, y, z)$.

(a) Find

$$\nabla\left(\frac{1}{r}\right) ,$$

where $r = |\boldsymbol{r}|$. [1]

(b) Prove that

$$\nabla \cdot \frac{\boldsymbol{r}}{r} = \frac{2}{r} .$$

[1]

10

You roll three fair 6-sided dice.

(a) What is the probability that they all roll 6? [1]

(b) What is the probability that at least one die rolls a 6? [1]

[It is sufficient to write the results as fractions.]

SECTION B

11S

Let X be a continuous real-valued random variable with probability density function

$$f(x) = \frac{\alpha}{x^{\alpha+1}}, \quad 1 < x < \infty$$

where $\alpha > 0$.

(a) Verify that $f(x)$ satisfies the normalisation condition. [3]

(b) Find the mean of X, and specify the range of values of α for which the mean exists. [3]

(c) Find the variance of X, and specify the range of values of α for which the variance exists. [4]

(d) Let $F(x)$ be the cumulative probability function of X. Find an expression for $F(x)$, and hence or otherwise, find the median. [6]

(e) Calculate the conditional probability $P(1 < X < 3 | 1 < X < 6)$. [4]

12W

(a) Write the following expressions in vector notation:

(i) $a_i b_k b_i a_k$ [1]

(ii) $a_i b_k \delta_{km} \delta_{im} + a_i b_k a_j b_k \delta_{ij}$ [2]

(iii) $(a_i a_i)^{\delta_{jj} + \delta_{kk}}$ [2]

where a_i and b_i are the components of three-dimensional vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively, δ_{ij} is the Kronecker delta and summation is assumed over repeated indices.

(b) Let $\boldsymbol{A}$ be an antisymmetric 2×2 matrix.

(i) Find the trace of matrix $\boldsymbol{A}$. [1]

Given that $\det(\boldsymbol{A}) = 1/4$ and $[\boldsymbol{A}]_{12} > 0$ find all elements of the following matrices:

(ii) $\boldsymbol{A}$ [1]

(iii) $\boldsymbol{A}^2$ and thus $\boldsymbol{A}^{-1}$ [2]

(iv) $\boldsymbol{A}^3$ [1]

(v) $(\boldsymbol{A})^{2n}$ and $(\boldsymbol{A})^{2n+1}$, where n is a positive integer. [4]

(vi) Demonstrate that

$$\sum_{n=1}^{\infty} \frac{\boldsymbol{A}^{4n} - 2\boldsymbol{A}^{2n}}{n} = \begin{pmatrix} \ln(5/3) & 0 \\ 0 & \ln(5/3) \end{pmatrix} .$$

[6]

13X

You may assume $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.

(a) Evaluate $\int_0^{\infty} e^{-x^2/4} dx$. [2]

(b) Define $I_n = \int_{-\infty}^{\infty} x^n e^{-x^2} dx$ for positive integer n.

(i) For $n \geqslant 2$ find I_n in terms of I_{n-2}. [3]

(ii) Evaluate I_1 and I_2. [2]

(c) (i) Without evaluating the integrals show that

$$\int_1^{\infty} \frac{1}{x^3} \ln x \, dx = -\int_0^1 x \ln x \, dx.$$

[3]

(ii) Evaluate

$$\int_0^{\infty} \frac{1}{1+x^2} \ln x \, dx.$$

[3]

(d) Show that

$$\int_0^{\pi/2} \frac{dx}{2+\sin x} = \int_0^1 \frac{dt}{t^2+t+1},$$

where $t = \tan(x/2)$, [3]

and evaluate the integral. [4]

14Z

(a) Find the general solution of

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} = 3x^2 + 2.$$

[6]

(b) Find the general solutions of the equations

$$\frac{d^2y}{dx^2} + 4y = \sin x$$

and

$$\frac{d^2y}{dx^2} + 4y = \sin(2x).$$

Hence find the general solution of

$$\frac{d^2y}{dx^2} + 4y = \sin x + \sin(2x).$$

[7]

(c) Show that the set of coupled equations

$$\frac{dy}{dt} = -y + 3x$$
$$\frac{dx}{dt} = 4x - 2y$$

can be reduced to a single second order differential equation for $y(t)$ and solve this equation subject to the initial conditions $y(0) = 2$ and $x(0) = 1$. [7]

15T

Consider the curve defined parametrically by

$$\begin{cases} x(t) &= a\cosh t \\ y(t) &= b\sinh t \end{cases}$$

where a and b are positive constants and $-\infty < t < \infty$.

(a) Write the expression for the distance $D(x,y)$ from an arbitrary point (x,y) on the curve to the origin and use it to find the coordinates (x_m, y_m) of the point on this curve at which this distance is minimal. [5]

(b) Write down the equation of the curve in Cartesian coordinates. [4]

(c) Derive the equations of the asymptotes of the curve and sketch the curve. [5]

(d) Consider the distances from an arbitrary point on the curve to the points $(-\sqrt{a^2+b^2}, 0)$ and $(\sqrt{a^2+b^2}, 0)$. Show that the difference between these distances is the same for all points on the curve. [6]

16R

(a) Consider a scalar field $V(x,y,z) = 2x^3 + y^2 + 3z^2 - y$.

(i) Find the gradient of V. [2]

(ii) Find the unit vector $\hat{\boldsymbol{n}}$ normal to the cylinder $x^2+y^2 = 8$ at the point $(2,2,-1)$. [2]

(iii) Find the rate of change of V in the direction of $\hat{\boldsymbol{n}}$ at this point. [2]

(iv) Let $\boldsymbol{\Gamma}$ be the intersection of the cylinder with the plane $z = 3$. Using the azimuthal angle as a parameter, calculate the line integral of $V(x,y,z)$ along $\boldsymbol{\Gamma}$ traversed anticlockwise. [8]

(b) Two vector fields $\boldsymbol{F}$ and $\boldsymbol{G}$ satisfy the relation $\boldsymbol{G} = \boldsymbol{\nabla} \times \boldsymbol{F}$. Prove that $\boldsymbol{\nabla} \cdot \boldsymbol{G} = 0$. [6]

17Y

[You must show your working in all parts of this question.]

(a) The first law of thermodynamics may be written as

$$dU = TdS - PdV$$

where U represents internal energy, P the pressure, V the volume, T the temperature and S the entropy. Given that dU is an exact differential, deduce the (Maxwell) relation

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial P}{\partial S}\right)_V .$$

[3]

There are three additional convenient thermodynamic potentials which correspond to exact differentials:

$F = U - TS$

$G = U + PV - TS$

$H = U + PV.$

From these, derive three more Maxwell relations, in addition to the relation given above, which relate the partial derivatives of the four quantities P, S, T, V. [9]

(b) Use the first law of thermodynamics as quoted above and Maxwell relations to prove that

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P. \qquad [\$]$$

In a van der Waals gas P is observed to obey

$$P = \frac{RT}{V-b} - \frac{a}{V^2}$$

where a, b, R are constants. Use equation [$] above to derive the general expression for $U(V,T)$ by requiring $U = cT$, where c is a constant, in the limit $V \to \infty$. [8]

18T

(a) Consider a curve in the (x, y) plane defined by the equation $F(x, y) = 0$. Show that the slope of the curve dy/dx is given by

$$-\frac{\partial F}{\partial x}\Big/\frac{\partial F}{\partial y}.$$

[5]

(b) A singular point of a curve is a point on the curve for which

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0.$$

Find the singular point of the curve defined by

$$(y - x^2)^2 - x^5 = 0. \qquad [\$\$]$$

[4]

(c) Determine the range of possible values of x and give the explicit expression $y = y(x)$ for each branch of the curve defined by [$$]. [4]

(d) Find x-coordinates for the points of the maxima, minima, inflexion and zeros (if any) for each branch $y = y(x)$ of the curve defined by [$$]. [4]

(e) Sketch each branch of this curve indicating clearly the points found in (d). [3]

19R*

(a) Consider a vector field $\boldsymbol{F}$. Write down the relation between the integral of the curl of $\boldsymbol{F}$ over a surface S and the line integral of $\boldsymbol{F}$ around the boundary C of S (Stokes's Theorem). With the aid of a diagram, explain the meaning of the notation used. [4]

(b) Consider the vector field $\boldsymbol{F}(x, y, z) = (0, 3xy - x, 2xz)$. Calculate the surface integral

$$\int_S (\boldsymbol{\nabla} \times \boldsymbol{F}) \cdot d\boldsymbol{S},$$

where S is taken over the open hemisphere $x^2 + y^2 + z^2 = a^2$ above the plane $z = 0$ and $d\boldsymbol{S}$ is the outward normal. [16]

20Z*

(a) Write down the derivative with respect to θ of the integral

$$\int_{a(\theta)}^{b(\theta)} f(x,\theta)\,dx.$$

[2]

(b) By differentiating the integral

$$\int_0^1 x^m\,dx,\ \ (m > -1)$$

n times, show that

$$\int_0^1 x^m (\ln x)^n\,dx = \frac{(-1)^n n!}{(m+1)^{n+1}}.$$

[7]

(c) The function $F(t)$ is given by

$$F(t) = \int_0^\infty \cos(tx)\,e^{-x^2/2}\,dx.$$

Show that

$$\frac{dF}{dt} = -tF(t),$$

and solve this equation for $F(t)$.

[You may use $\int_0^\infty e^{-x^2/2} dx = \sqrt{\pi/2}$.] [11]

END OF PAPER