UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS **Part IB & II (General)**

Tuesday, 29 May, 2012 9:00 am to 12:00 pm

MATHEMATICS (1)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the left hand margin.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, ***6A****).*

Answers must be tied up in ***separate*** *bundles, marked* ***A, B or C*** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A ***separate*** *green master cover sheet listing all the questions attempted* ***must*** *also be completed.*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Green master cover sheet
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1C

Let $\mathbf{F}$ be a vector field and $\mathbf{a}$ be an arbitrary constant vector.

Show that

$$\nabla \times (\mathbf{a} \times \mathbf{F}) = \mathbf{a}\,(\nabla \cdot \mathbf{F}) - (\mathbf{a} \cdot \nabla)\mathbf{F}\,.$$

[4]

State Stokes' theorem. [2]

By applying Stokes' theorem to the above identity, show that

$$\int_C d\mathbf{l} \times \mathbf{F} = \int_S (d\mathbf{S} \times \nabla) \times \mathbf{F}$$

for any closed curve C that bounds a surface S. [8]

Verify this identity for the case where C is a square path starting at $(0,0,0)$, then progressing in a straight line to $(0,1,0)$, then to $(1,1,0)$, then to $(1,0,0)$ and finally back to the origin, with $\mathbf{F} = \mathbf{r}$ and $\mathbf{r} = (x,y,z)$. [6]

2C

The number density of neutrons $n(\mathbf{r},t)$ in a lump of uranium is determined by the partial differential equation

$$\nabla^2 n = \frac{\partial n}{\partial t} - \lambda n$$

where ∇^2 is the Laplacian operator, $\mathbf{r}$ is the position vector, t is the time and λ is a constant.

Suppose the lump of uranium is a sphere of radius a and the density of neutrons is spherically symmetric. Furthermore, suppose this equation can be solved by the method of separation of variables so that

$$n = R(r)T(t)$$

where r is the distance from the centre of the sphere. Find two ordinary differential equations for $R(r)$ and $T(t)$. [4]

Suppose that the density of neutrons is never zero except at the surface of the sphere and finite everywhere inside. Find $n(\mathbf{r},t)$. [8]

Show that the concentration of neutrons will grow as a function of time provided that

$$\lambda > \frac{\pi^2}{a^2}\,.$$

[8]

[*Hint: To find $R(r)$, make the substitution $R(r) = r^p f(r)$ for some p.*]

3C

Find the general solution $y(x)$ to the homogeneous second-order linear differential equation

$$\frac{d^2y}{dx^2} + \frac{3}{x}\frac{dy}{dx} + \frac{y}{x^2} = 0\,.$$

[6]

Construct the Green's function for this equation in the region $0 \leqslant x < \infty$, which satisfies

$$\frac{d^2G(x,\xi)}{dx^2} + \frac{3}{x}\frac{dG(x,\xi)}{dx} + \frac{G(x,\xi)}{x^2} = \delta(x-\xi)$$

subject to the boundary conditions $G(0,\xi) = \frac{dG(0,\xi)}{dx} = 0$, where $\delta(x-\xi)$ is the Dirac delta function. [8]

Use your Green's function to solve the differential equation

$$\frac{d^2y}{dx^2} + \frac{3}{x}\frac{dy}{dx} + \frac{y}{x^2} = x$$

for $x \geqslant 0$, subject to the boundary conditions $y(0) = y'(0) = 0$. [6]

4C

Calculate the Fourier transform of the function

$$f(x) = e^{-\lambda x^2} \, ,$$

where λ is a positive constant. [5]

Consider the partial differential equation for $\psi(x,t)$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial \psi}{\partial t}.$$

Find the ordinary differential equation that

$$\tilde{\psi}(k,t) = \int_{-\infty}^{\infty} \psi(x,t) \; e^{-ikx} \; dx$$

obeys. [5]

Find $\tilde{\psi}(k,t)$ given $\psi(x,0) = e^{-\lambda x^2}$. [5]

Hence find $\psi(x,t)$ for $t > 0$. [5]

[*Hint: You may use the following relation*

$$\int_{-\infty}^{\infty} e^{-\lambda(x+i\alpha)^2} \; dx = \sqrt{\frac{\pi}{\lambda}}$$

for α *and* λ *real constants and* $\lambda > 0$.]

5A

(i) Show that an $n \times n$ matrix A is diagonalisable if it has n linearly independent eigenvectors. Show that if A is diagonalisable, then so is $\mathsf{B} = \mathsf{P}^{-1}\mathsf{A}\mathsf{P}$, where P is an $n \times n$ matrix and P^{-1} is its inverse. [6]

(ii) Let λ_i (with $i = 1, 2, \ldots, n$) be the eigenvalues of an $n \times n$ Hermitian matrix A.

(a) Show that

$$\mathrm{Tr}\,\mathsf{A}^k = \sum_{i=1}^{n} \lambda_i^k \qquad \text{and} \qquad \det \mathsf{A}^k = \prod_{i=1}^{n} \lambda_i^k \ ,$$

for all positive integers k. [6]

(b) Show that

$$\det(\exp \mathsf{A}) = \exp(\mathrm{Tr}\,\mathsf{A}) \ .$$

[4]

(c) If $\mathsf{A}^2 = \mathsf{A}$, prove that either (i) $\det \mathsf{A} = 1$ and $\mathrm{Tr}\,\mathsf{A} = n$ or (ii) $\det \mathsf{A} = 0$ and $\mathrm{Tr}\,\mathsf{A} = m < n$, where m is an integer. [4]

6A

(i) Let A be a complex $n \times n$ matrix, and define

$$\mathsf{H} \equiv \frac{\mathsf{A} + \mathsf{A}^\dagger}{2} \quad \text{and} \quad \mathsf{S} \equiv \frac{\mathsf{A} - \mathsf{A}^\dagger}{2i} \ .$$

Let λ be an eigenvalue of A and $\mathbf{x}$ be the corresponding unit-normalised eigenvector.

(a) Show that $\lambda = \mathbf{x}^\dagger \mathsf{H} \mathbf{x} + i\mathbf{x}^\dagger \mathsf{S} \mathbf{x}$. [4]

(b) Show that the real part of λ is given by $\mathrm{Re}(\lambda) = \mathbf{x}^\dagger \mathsf{H} \mathbf{x}$ and the imaginary part is given by $\mathrm{Im}(\lambda) = \mathbf{x}^\dagger \mathsf{S} \mathbf{x}$. [4]

(ii) Let B be a Hermitian $n \times n$ matrix with n distinct (real) eigenvalues λ_i.

(a) Show that the corresponding normalised eigenvectors $\mathbf{x}_i$ satisfy

$$\mathbf{x}_i^\dagger \mathbf{x}_j = \delta_{ij} \ ,$$

and therefore form an orthonormal basis, $\{\mathbf{x}_i \ : \ i = 1, 2, \cdots, n\}$. [6]

(b) Now, consider

$$\beta \equiv (\mathsf{B}\mathbf{v} - a\mathbf{v})^\dagger (\mathsf{B}\mathbf{v} - b\mathbf{v}) \ ,$$

where $\mathbf{v}$ is an arbitrary vector, while a and b are real constants with $a < b$. By expanding $\mathbf{v}$ in terms of the eigenvectors $\mathbf{x}_i$, show that $\beta > 0$ if no eigenvalue lies in the interval $[a, b]$. [6]

7A

(i) Two-dimensional fluid flow can be described by a complex potential

$$f(z) = u(x,y) + iv(x,y) \ ,$$

where $z \equiv x + iy$. Let the fluid velocity be $\mathbf{V} = \nabla u$. If $f(z)$ is analytic, show that

(a) $\nabla \cdot \mathbf{V} = 0,$ [4]

(b) $\frac{df}{dz} = V_x - iV_y.$ [4]

[*You may assume the Cauchy-Riemann equations without proof.*]

(ii) Consider the Gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \quad \text{for} \quad \mathrm{Re}(z) > 0.$$

(a) Using integration by parts, derive the recursion relation

$$\Gamma(z+1) = z\,\Gamma(z) \ . \qquad (\star)$$

Also show that $\Gamma(1) = 1$. [4]

(b) Assuming that $(\star)$ holds for all $z \in \mathbb{C}$, show that $\Gamma(z)$ has simple poles at all non-positive integers. Compute their residues. [8]

8A

Legendre's equation

$$(1-x^2)y'' - 2xy' + \ell(\ell+1)y = 0 \ ,$$

admits series solutions of the form $y = \sum_{n=0}^{\infty} a_n x^n$.

(a) Derive the recurrence relation for a_n. [4]

(b) Show that for integer ℓ, one of the solutions, P_ℓ, is a polynomial of order ℓ; while the other solution is an infinite series Q_ℓ. [2]

(c) Find the first four polynomials $P_\ell(x)$ (i.e. $\ell = 0, 1, 2, 3$) given the normalisation $P_\ell(1) = 1$. [4]

(d) Show that the Wronskian of P_ℓ and Q_ℓ is given by

$$P_\ell Q_\ell' - P_\ell' Q_\ell = \frac{A_\ell}{1-x^2} \ ,$$

for A_ℓ independent of x. [6]

(e) Derive $Q_0(x)$ in closed form, assuming $Q_0(0) = 0$. [4]

9A

(i) (a) State the Euler-Lagrange equation corresponding to stationary values of the functional

$$I[y(x)] = \int_a^b f(x, y(x), y'(x))\,\mathrm{d}x \ ,$$

for fixed $y(a)$ and $y(b)$. [2]

(b) Derive the first integral of the Euler-Lagrange equation for the case where f is independent of $y(x)$. [3]

(ii) Driving on a hot asphalt road, you may see the road in the distance appear to be covered by what looks like water. This mirage effect arises because the refractive index of air depends on temperature and is smaller near the surface of the hot road.

(a) Let x be the height above the road and y be a coordinate along the road. The travel time of light is the following functional of the path taken

$$\int n(x)\sqrt{1+(y')^2}\,\mathrm{d}x \ ,$$

where $n(x)$ is the refractive index.
Show that the path of least time satisfies

$$y' = \frac{dy}{dx} = \frac{c}{\sqrt{n(x)^2 - c^2}} \ , \qquad (\star)$$

where c is a real constant. [4]

(b) Now let $n(x) = 1 + \beta x$, where $\beta > 0$ is a real constant.
By integrating $(\star)$, show that

$$x = -\frac{1}{\beta} + \frac{c}{\beta}\cosh\left[\frac{\beta}{c}(y - y_0)\right] \ ,$$

where the integration constant is defined such that $y_0 = y(x_0)$ at $x_0 = (-1 + c)/\beta$. [8]

(c) For $c > 1$, sketch the path of the light $x(y)$. [3]

10A

(i) Consider the Sturm-Liouville equation

$$-\frac{d}{dx}\left(p(x)\frac{d\psi}{dx}\right) + q(x)\psi = \lambda w(x)\psi \ ,$$

where $p(x) > 0$ and $w(x) > 0$ for $\alpha < x < \beta$.

(a) Show that finding the eigenvalues λ is equivalent to finding the stationary values of the functional

$$\Lambda[\psi(x)] = \int_\alpha^\beta (p\psi'^2 + q\psi^2)\mathrm{d}x \ ,$$

subject to the constraint

$$\int_\alpha^\beta w\psi^2 \,\mathrm{d}x = 1 \ .$$

You may assume that $\psi(x)$ satisfies suitable boundary conditions at $x = \alpha$ and $x = \beta$ (which should be stated). [6]

(b) Explain briefly the Rayleigh-Ritz method for estimating the lowest eigenvalue λ_0. [4]

(ii) The wavefunction $\psi(x)$ for a quantum harmonic oscillator satisfies

$$\left[-\frac{d^2}{dx^2} + x^2\right]\psi = \lambda\psi \ . \qquad (\star)$$

(a) Use the trial function

$$\psi(x) = \begin{cases} \sqrt{\frac{15}{16a^5}}(a^2 - x^2) & \text{for } |x| \leqslant a \\ 0 & \text{for } |x| > a \end{cases}$$

to estimate the lowest eigenvalue λ_0. [8]

(b) The exact ground state wavefunction is

$$\psi_0(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2} \ .$$

Find the corresponding eigenvalue and compare it to the previous estimate. [2]

END OF PAPER