UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Monday, 11 June, 2012 9:00 am to 12:00 pm

MATHEMATICS (1)

Before you begin read these instructions carefully:

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

You may submit answers to ***all*** *of section A, and to no more than* ***five*** *questions from section B.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on one side of the paper only and begin each answer on a separate sheet. *(For this purpose, your section A attempts should be considered as one single answer.)*

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Tie up ***all of your section A answer*** *in a single bundle, with a completed blue cover sheet.*

Each section B question has a number and a letter (for example, ***11X****). Answers to each question must be tied up in* ***separate*** *bundles and marked (for example* ***11X, 12Z*** *etc) according to the number and letter affixed to each question.* ***Do not join the bundles together****. For each bundle, a blue cover sheet* ***must*** *be completed and attached to the bundle, with the correct number and letter written in the section box.*

A ***separate*** *green master cover sheet listing all the questions attempted* ***must*** *also be completed. (Your section A answer may be recorded just as A: there is no need to list each individual short question.)*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Green master cover sheet
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1

Write $f(x) = 3x^2 - 12x + 11$ in the form $a(x+b)^2 + c$.

State the minimum value of $f(x)$ and the value of x where this occurs. [2]

2

A curve has the equation $y = 2x + x^2$. Find the equation of the normal to the curve passing through the point $(1, 3)$. [2]

3

The function $f(x)$ is given by $f(x) = 2x^3 - px^2 - px + 21$. When $f(x)$ is divided by $(x-1)$ the remainder is 9. Find p. [1]

Divide $(x^3 + x^2 + 2x + 1)$ by $(x-1)$ giving quotient and remainder. [1]

4

Differentiate with respect to x

(a) xe^{-2x}, [1]

(b) $\frac{\ln x}{x}$. [1]

5

Calculate the integral

$$\int_0^\pi x^2 \sin x \, dx.$$

[2]

6

Express $\dfrac{4x^2 - x}{(x-1)^2(x+2)}$ as partial fractions. [2]

7

Solve the differential equation

$$\frac{dy}{dx} = 3x^2 y \ ,$$

given that $y = 3$ when $x = 1$. [2]

8

(a) Evaluate $\displaystyle\sum_{n=4}^{\infty} \left(\frac{1}{2}\right)^n$. [1]

(b) Calculate the term in x^3 in the expansion of $(1 + x)^9$. [1]

9

Sketch the graphs of

(a) $\dfrac{1}{e^x - 1}$ for $-\infty < x < \infty$, [1]

(b) $e^{-x}|\sin x|$ for $0 \leqslant x \leqslant 4\pi$. [1]

10

Find the volume of the solid generated by rotating the part of the curve $y = \sin x$ between $x = 0$ and $x = \pi$ about the x-axis by 2π. [2]

SECTION B

11X

(a) Express the following system of simultaneous equations in the matrix form $\boldsymbol{A}\boldsymbol{r} = 0$, where $\boldsymbol{r} = (x, y, z)^T$:

$$\begin{aligned} ax + 2z &= 0 \\ (b+1)y + (c-1)z &= 0 \\ (c+1)y + (b-1)z &= 0. \end{aligned}$$

Find conditions on a, b and c such that there exist non-trivial solutions. [5]

(b) Solve the system for the special case $a = 1$, $b = 2$ and $c = 2$. [5]

(c) Find the values of a, b and c such that the vector $\boldsymbol{A}\boldsymbol{r}$ is perpendicular to the vector $(1, 2, 1)$ for all choices of $\boldsymbol{r}$ excluding the case $\boldsymbol{r} = \boldsymbol{0}$. [5]

(d) Is it possible to find values of a, b and c such that the vector $\boldsymbol{A}\boldsymbol{r}$ is perpendicular to the vector $(1, -1, 1)$ for all choices of $\boldsymbol{r}$ excluding the case $\boldsymbol{r} = \boldsymbol{0}$? Explain your reasoning. [5]

12Z

(a) Find the real and imaginary parts of

(i) $\left(\frac{i+2}{i-1}\right)^2$, [2]

(ii) $(1+i)^{10}$, [2]

(iii) $\sin\left(\frac{\pi}{2} + i \ln 2\right)$. [4]

(b) Find the square roots of $(i - 1)$. [2]

(c) Express $\sin^5 \theta$ as a linear combination of sines of multiples of θ. [5]

(d) Find the solutions of the equation $\cosh z = -1$. [5]

13Z

(a) Solve the equation

$$\frac{dy}{dx} = \frac{y^2 + xy}{x^2}.$$

[6]

(b) Show that

$$(x + y)\, dx + x\, dy$$

is an exact differential, and use this to obtain the general solution of

$$x\,\frac{dy}{dx} + x + y = 0.$$

[7]

(c) Solve the equation

$$\frac{dy}{dx} + ky = a \sin mx$$

subject to the boundary condition $y = 1$ when $x = 0$, where k, m and a are real, non-zero, constants. [7]

14S

(a) Write down an expansion of the vector $\boldsymbol{a} \wedge (\boldsymbol{b} \wedge \boldsymbol{c})$ as a linear combination of $\boldsymbol{b}$ and $\boldsymbol{c}$.

[Notation: $\boldsymbol{a} \wedge \boldsymbol{b}$ is equivalent to $\boldsymbol{a} \times \boldsymbol{b}$.] [4]

(b) Show that a necessary condition for the planes

$$\boldsymbol{r} \cdot \boldsymbol{b} = \lambda, \qquad \boldsymbol{r} \cdot \boldsymbol{c} = \mu$$

to intersect in a line is

$$\boldsymbol{b} \wedge \boldsymbol{c} \neq \boldsymbol{0},$$

where $\boldsymbol{b}$ and $\boldsymbol{c}$ are non-zero vectors, and λ and μ are scalars.

Give a geometrical interpretation of the case where $\boldsymbol{b} \wedge \boldsymbol{c} = \boldsymbol{0}$. [4]

(c) Assume that $\boldsymbol{b} \wedge \boldsymbol{c} \neq \boldsymbol{0}$. A point that lies on the line of intersection of the two planes has position vector $\boldsymbol{r}$. Show that $\boldsymbol{r}$ satisfies

$$\boldsymbol{r} \wedge (\boldsymbol{b} \wedge \boldsymbol{c}) = \mu \boldsymbol{b} - \lambda \boldsymbol{c} \; .$$

Hence deduce that the two planes intersect in a straight line whose equation may be written in the form

$$\boldsymbol{r} = \boldsymbol{a} + t(\boldsymbol{b} \wedge \boldsymbol{c}),$$

where $\boldsymbol{a}$ is a fixed vector and t is a scalar parameter. Write down the conditions that $\boldsymbol{a}$ must satisfy. [4]

(d) The vector $\boldsymbol{a}$ is chosen so that the point on the line of intersection closest to the origin has $t = 0$. Show in this case that $\boldsymbol{a}$ satisfies

$$\boldsymbol{a} \cdot (\boldsymbol{b} \wedge \boldsymbol{c}) = 0 \; .$$

Hence deduce that $\boldsymbol{a}$ can be written in the form

$$\boldsymbol{a} = \alpha \boldsymbol{b} + \beta \boldsymbol{c} \; ,$$

for some scalars α and β. [4]

(e) In the case where $\boldsymbol{b} \cdot \boldsymbol{c} = 0$ show that

$$\alpha = \frac{\lambda}{|\boldsymbol{b}|^2}, \qquad \beta = \frac{\mu}{|\boldsymbol{c}|^2},$$

and hence calculate the closest distance of approach to the origin of the line of intersection of the planes. [4]

15T

A cardioid is a curve described in plane-polar coordinates (r,θ) by the equation $r = 1 + \cos\theta$, for $0 \leqslant \theta < 2\pi$.

(a) Sketch the curve in the (x,y) plane and find the Cartesian coordinates (x_M, y_M) of the point of maximum y. [6]

(b) Calculate the area inside the curve. [7]

(c) Show that

$$(dx)^2 + (dy)^2 = (dr)^2 + (r d\theta)^2$$

and hence calculate the length of Γ, the perimeter of the cardioid, given by

$$\int_\Gamma \sqrt{(dx)^2 + (dy)^2}.$$

[7]

16Y

(a) Find the rate of change (directional derivative) of

$$f(x,y) = \frac{1}{1 + x^2 + y^2}$$

at the point (1,0) in the direction of the vector $\boldsymbol{v} = 4\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}}$. [8]

(b) Show that the function

$$g(x,y) = (x^2 - y^2)\, e^{-(x^2+y^2)/2}$$

has a stationary point at $(0,0)$ and find the other stationary points. [6]

(c) Classify, showing method, the stationary point of $g(x,y)$ at $(0,0)$. [6]

17R

(a) Determine

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx$$

for all values of integers $m \geqslant 0$, $n \geqslant 0$. [8]

(b) Find the Fourier series of the function $f(x) = e^x$ on the interval $-1 < x < 1$, which is defined by periodicity 2 outside that interval. [12]

18R

(a) Ben is attending a 24 lecture course. Every morning Ben tosses a biased coin with probability 3/4 and 1/4 for heads and tails, respectively. Ben goes to the lecture if the coin lands heads. Otherwise, he stays at home.

(i) Write down the probability of Ben attending all 24 lectures. [1]

(ii) Write down the probability of Ben attending none of the lectures. [1]

(iii) Find the expectation value of the number of lectures Ben attends. [2]

(iv) Find the standard deviation of the number of lectures Ben attends. [2]

(b) Let X be a continuous real-valued random variable with probability distribution $f(x)$.

(i) What is the probability of X having a value in the range $[\alpha, \beta]$? [1]

(ii) Define the mean, μ, and variance, σ^2, of X. [2]

(c) Consider the following probability distribution

$$f(x) = \begin{cases} Axe^{-\lambda x} & \text{for } x \geqslant 0 \\ 0 & \text{for } x < 0, \end{cases}$$

where A and λ are positive constants.

(i) Find the value of A in terms of λ. [2]

(ii) Find the mean and variance of X. [4]

(iii) Show that the probability that X takes a value in excess of two standard deviations from the mean is given by

$$P(X > \mu + 2\sigma) = (3 + 2\sqrt{2})e^{-2(1+\sqrt{2})}$$

[5]

19W*

(a) (i) A real function $f(x)$ is defined on an interval containing the point x_0. Explain what is meant by the statement that the function $f(x)$ is (1) continuous, (2) differentiable at $x = x_0$. [2]

(ii) Investigate if the function

$$f(x) = \begin{cases} \exp\left(-1/x^2\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

is continuous and differentiable at $x = 0$. [4]

(iii) Find $\lim_{x\to 0} d^n f(x)/dx^n$ for an arbitrary order of differentiation n. Explain your answer. [2]

(iv) For $x \neq 0$, find a series expansion of $f(x)$ in powers of $1/x^2$, that is, find the coefficients a_n in the expression

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{x^{2n}}.$$

[2]

(v) Sketch $f(x)$ and $df(x)/dx$.
[The coordinates of the stationary points of $df(x)/dx$ are not required.] [4]

(b) Let y_n be an infinite sequence of real numbers obeying the recurrence relation

$$y_{n+1} = \sqrt{2 + y_n},$$

where $n = 0, 1, 2, \ldots$, and y_0 is an arbitrary non-negative value.

(i) Find a value y_* of y_0 for which all the members of the sequence are the same. [2]

(ii) Demonstrate graphically that for any $0 \leqslant y_0 < y_*$ the sequence $(y_0, y_1, y_2, \ldots)$ approaches a limit as $n \to \infty$ and find this limiting value. Undertake a similar analysis for $y_0 > y_*$.
[*Hint: You might find it helpful to sketch the functions* $g_1(x) = x$ *and* $g_2(x) = \sqrt{2 + x}$.] [4]

20Y*

(a) A rectangular box without a lid has length l, width w and height h. The external surface area of the box is required to be $12\,\text{m}^2$. Using the method of Lagrange multipliers, find the dimensions that maximize the volume of the box. [10]

(b) (i) Sketch the curve $x^2 + xy = 1$. [3]

(ii) Using the method of Lagrange multipliers, determine the point on the curve $x^2 + xy = 1$ that is nearest to the origin (0,0) and has $x > 0$. [7]

END OF PAPER